Inference for stochastic bubble trend in stock price under error correction model

by

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Abstract

The paper suggests there may be an I(1) stochastic bubble trend in the stock price even if the I(1) stock price and dividend are cointegrated that are usually confirmed in the empirical tests. For this, we show the long run equilibrium of stock price may be decomposed of fundamental and bubble stochastic trends; i.e., the sum of dividend innovations and the sum of innovations that are orthogonal with the dividend innovations, through the Beveridge Nelson decomposition and projection. Under this VAR construction, there is an error correction mechanism where the stock price converges to its long run equilibrium which includes the stated stochastic bubble trend. In application for the US monthly data 1871.1-2010.9, the fluctuation of stock price was mostly explained not by the stochastic trend of dividend shocks but by the stochastic bubble trend confirming the findings of Shiller-LeRoy and Porter.

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1 Introduction

Recent global financial crisis invoked the importance to check the bubble in asset prices dramatically. Historically Shiller (1981), LeRoy and Porter (1981) show that stock prices are too volatile to be attributed to market fundamentals failure of simple present value model: discounted value of future dividends. Variable discount specification also has been introduced to relax a constant discount rate while maintaining the notion that stock prices are exclusively determined by the market fundamental e.g., Campbell and Shiller(1988), West(1987). Contrastively, intrinsic bubble has been introduced by Froot and Obstfeld (1991), where the bubble is a function of fundamental. These models typically assume the self fulfilling expectations generated by extraneous events or rumors driven by self fulfilling expectations.

The concept of rational bubble in essence focuses on the discrepancy between the stock price and fundamental dividend may exist6 in the long run. However the rational bubble is explosive and thus the stock price also explosive ex post that seems to be rare from the theoretical and empirical perspective. Tirole (1982) has shown that rational bubbles can not arise in a model with a fixed finite number of representative agents, and Tirole (1985) has also shown that in nonstochastic overlapping-agenertion models bubbles are ruled out if the interest rate is grater than the growth rate. A related condition is cited in O'Connel and Zeldes (1988) for representative-agents models with a growing population. Evans (1989) provides theoretical reasons for doubting that bubbles, in rational-expectations models, can arise as the outcomes of a learning process.

Diba and Grossman (1984, 1988a), argue that if stock prices are not more explosive than dividends then it can be concluded that rational bubbles are not present, since they would generate an explosive component to stock prices. Using unit-root tests and cointegration tests, Diba and Grossman (1988b p. 529) state that "the analysis supports the conclusion that stock prices do not contain explosive rational bubbles." Evans (1991 p. 922) shows the above battery of tests is unable to detect an important class of rational bubbles like periodically collapsing bubbles. Wu (1997) considered the possibly negative bubble process where the bubble is treated as an unobserved state vector which is estimated by the Kalman filtering. He assumes that the shock in the parametric bubble process is not correlated with the dividend innovation. However this assumption needs to be tested rather than to suppose it. Recently Engsted (2006) analyze the explosive bubble in the context of Johansen's cointegrated VAR model based upon Nielsen (2010). The situation of main interest in Nielsen (2010) is when there are both unit roots and a single explosive root present.

This paper suggests that there may be an I(1) stochastic bubble trend in the stock price even if the I(1) stock price and dividend are cointegrated that are usually confirmed in the

empirical tests. For this, we show the long run equilibrium of stock price may be decomposed of fundamental and bubble stochastic trends; i.e., the sum of dividend innovations and the sum of innovations that are orthogonal with the dividend innovations, through the Beveridge Nelson decomposition and projection. Under this VAR construction, there is an error correction mechanism where the stock price converges to its long run equilibrium which includes the stated stochastic bubble trend. In application for the US monthly data 1871.1-2010.9, the fluctuation of stock price was mostly explained not by the stochastic trend of dividend shocks but by the stochastic bubble trend confirming the findings of Shiller (1981) and LeRoy and Porter (1981).

We may summarize that the SBT is different from rational bubble estimated in Wu (1997) in two respects. First, the rational bubble is an explosive process while the SBT is an I(1) process, which exists even under the error correction model assuming a cointegration between the stock price and dividend. Second, the rational bubble is estimated under an orthogonality condition for the Kalman filtering while the SBT does not require such a restriction. However the SBT has an important similarity with the rational bubble: i.e., the stock price includes the bubble in the long run.

The rest of this paper proceeds as follows. Section 2 discusses on the structural VAR interpretation of rational bubble model. Section 3 is on the estimation and test of SBT. Section 4 is on the application for the United States stock price and Section 6 concludes. In particular, we use $\xrightarrow{p}$ and $\xrightarrow{d}$ to signify convergence in probability and distribution, respectively.

2 Structural VAR interpretation of rational bubble model

Consider the standard linear rational expectations model of stock price determination,

$$E_t \left[P_{t+1} + D_t \right] / P_t = 1 + r \tag{1}$$

where:

P_t = the real stock price at time t;

D_t= the real dividend paid at time t;

E_t = the conditional expectation on information available at time t; and

r = the required real rate of return, $r > 0$.

Equation (1) is the arbitrage condition which states that the expected return from holding stocks should be equal to the required rate of return. According to Wu (1997), one weakness of the specification in levels is that a negative bubble today implies that there is a positive probability that at some future time, the bubble will be large and sufficiently

negative to make the stock price negative. To avoid the problem of obtaining negative theoretical prices, Wu (1997) expresses the present value model in terms of the natural logarithms of price and dividend as follows:

$$q = k + \rho E_t p_{t+1} + (1-\rho)\, d_t - p_t \tag{2}$$

where:

$q =$ the required log gross return rate;

$\rho =$ the average ratio of the stock price to the sum of the stock price and the dividend, $0 < \rho < 1$;

$k = -\ln(\rho) - (1-\rho)\ln(1/\rho - 1)$;

$p_t =$ln(P_t); and

$d_t =$ln(D_t).

Now the unique forward-looking, no-bubble solution to (2), denoted by p_t^f, is given by

$$p_t^f = (k-q)\,/\,(1-\rho) + (1-\rho)\textstyle\sum_{i=0}^{\infty} \rho^i E_t\,(d_{t+i}) \tag{3}$$

provided that the transversality condition is satisfied,

$$\lim_{i\to\infty} \rho^i E_t\,(p_{t+i}) = 0. \tag{4}$$

Equation (3) is the present value relation which states that the log stock price is equal to the present value of expected future log dividend streams. Notice that if the transvenality condition is violated, then (3) is only a particular solution to (2).

So the general solution to (2) takes the following form:

$$\begin{aligned} p_t &= (k-q)\,/\,(1-\rho) + (1-\rho)\textstyle\sum_{i=0}^{\infty} \rho^i E_t\,(d_{t+i}) + b_t \\ &= p_t^f + b_t \end{aligned} \tag{5}$$

where, $\{b_t\}$ satisfies the following homogeneous difference equation:

$$E_t\,(b_{t+i}) = (1/\rho)^i\, b_t \text{ for } i = 1, 2, ... \tag{6}$$

For instance if the log dividend grows with the constant g as $d_{t+1} = g d_t$ following Gordon (1959), then $p_t^f = (k-q)\,/\,(1-\rho) + (1-\rho)\,/\,(1-\rho g)\, d_t$ and

$$p_t = \tfrac{k-q}{1-\rho} + \tfrac{1-\rho}{1-\rho g} d_t + b_t. \tag{7}$$

In equation (5), the no-bubble solution is exclusively determined by dividends and is

often called the market-fundamental solution, while b_t, can be driven by events extraneous to the fundamental and is referred to as a rational bubble.

Wu (1997) assumes that the log dividends contain a unit root and can be approximated by an ARIMA(p,l,0) process as follows:

$$d_t = \mu + \sum_{i=1}^{p} \varphi_i d_{t-i} + \delta_t \tag{8}$$

where δ_t, is an i.i.d. error term.

Assuming that the bubble process $\{b_t\}$ is linear, equation (6) implies the following parametric form:

$$b_t = (1/\rho)\, b_{t-1} + \eta_t \tag{9}$$

or

$$b_t = \sum_{i=0}^{\infty} \rho^{-i} \eta_{t-i}$$

where the innovation η_t is assumed to be serially uncorrelated and have zero mean and finite variance σ_η^2. In Wu (1997), the bubble is treated as an unobserved state vector which is estimated by the Kalman filtering technique assuming η_t is uncorrelated with the dividend innovation, δ_t in equation (8) or $E\eta_t\delta_t = 0$.

First, I show the linear model in Wu (1997) represented by the equations (7), (8) and (9) is in fact a reduced form VAR model. Now we plug $d_t (= \mu + \sum_{i=1}^{p} \varphi_i d_{t-i} + \delta_t)$ and $b_t\, [= p_t - (k-q)\,/\,(1-\rho) - (1-\rho)\,/\,(1-\rho g)\, d_t]$ into the bubble process (9) to get following equation:

$$\begin{aligned} p_t \;=\; & \tfrac{(1-\rho)\mu}{1-\rho g} + \tfrac{q-k}{\rho(1-\rho)} + \tfrac{(1-\rho)(\varphi_1 - 1/\rho)}{1-\rho g} d_{t-1} + \tfrac{1}{\rho} p_{t-1} + \\ & \tfrac{1-\rho}{1-\rho g} \sum_{i=2}^{p} \varphi_i d_{t-i} + \tfrac{1-\rho}{1-\rho g} \delta_t + \eta_t \end{aligned} \tag{10}$$

after a simple arrangement. Now a reduced form VAR model of (8) and (10) is written as:

$$\begin{aligned} Z_t \;=\; & \begin{pmatrix} \mu \\ \frac{(1-\rho)\mu}{1-\rho g} + \frac{q-k}{\rho(1-\rho)} \end{pmatrix} + \begin{pmatrix} \varphi_1 & 0 \\ \frac{(1-\rho)(\varphi_1 - 1/\rho)}{1-\rho g} & 1/\rho \end{pmatrix} Z_{t-1} + \\ & \sum_{i=1}^{p} \begin{pmatrix} 1 & 0 \\ \frac{\rho-1}{1-\rho g} & 1 \end{pmatrix} \begin{pmatrix} \varphi_i & 0 \\ 0 & 0 \end{pmatrix} Z_{t-i} + \begin{pmatrix} \delta_t \\ \frac{1-\rho}{1-\rho g}\delta_t + \eta_t \end{pmatrix} \end{aligned} \tag{11}$$

where $Z_t = (d_t, p_t)'$.

Now above model (11) may be interpreted as a structural VAR model [henceforth SVAR]

as:

$$\begin{pmatrix} 1 & 0 \\ \frac{\rho-1}{1-\rho g} & 1 \end{pmatrix} Z_t = \begin{pmatrix} \mu \\ \frac{g-k}{\rho(1-\rho)} \end{pmatrix} + \begin{pmatrix} \varphi_1 & 0 \\ \frac{\rho-1}{\rho(1-\rho g)} & 1/\rho \end{pmatrix} Z_{t-1} + \sum_{i=1}^{p} \begin{pmatrix} \varphi_i & 0 \\ \frac{\varphi_i(\rho-1)}{1-\rho g} & 0 \end{pmatrix} Z_{t-i} + \begin{pmatrix} \delta_t \\ \eta_t \end{pmatrix}. \tag{12}$$

After influential work of Sims (1980), above triangular structure (12) and orthogonality assumption $E\eta_t\delta_t = 0$ are standard for the dynamic analyses including impulse response analysis and variance decomposition.

Two points are noteworthy for the improvement of SVAR model (12). First, the orthogonality assumption $E\eta_t\delta_t = 0$ is critical for the estimation of bubble. However this kind of orthogonality assumption is not imposed in the estimation procedure of SBT. Rather we attain it by the projection directly. Second, the stock price and its fundamental should be cointegrated when the fundamental theoretically explain the stock price. These are discussed more serious in following section.

3 VAR and MA representation of cointegration error and dividend change

In this section, we introduce a VAR and MA (moving average) representation of cointegration error and dividend change. It will let the estimation and test of SBT very conveniently. For this, at first, above model (11) is written as a generalized reduced form VAR model of order p as:

$$Z_t = \sum_{i=1}^{p} \Pi_i Z_{t-i} + \epsilon_t; \ t = 1, 2, ..., n \tag{13}$$

or

$$\Delta Z_t = \Phi Z_{t-1} + \sum_{i=1}^{p-1} \Phi_i \Delta Z_{t-i} + \epsilon_t \tag{14}$$

where Π_i is a 2×2 coefficient matrix for i=1,2,...p, Z_t is demeaned and detrended vector of $(d_t, p_t)'$, $\Delta Z_t = Z_t - Z_{t-1}$, $\bar{\Pi} = \sum_{i=1}^{p} \Pi_i$, $\Phi = \bar{\Pi} - I_2$ and $\epsilon_t = (\delta_t, \xi_t)'$ is a vector of i.i.d mean zero disturbances. Now we assume the stock price and dividend which are integrated of order one, are cointegrated represented by the standard singularity restriction [Johansen, 1991] of long run impact matrix Φ as:

Assumption 1 *Suppose* $\Phi = \alpha\beta$ *where* $\alpha = (\alpha_1, \alpha_2)$ *and* $\beta = (-\gamma, 1)$ with $\gamma \neq 0$.

Now the process (13) may be represented by the error correction model [henceforth

ECM] under Assumption 1 as:[2]

$$\Delta Z_t = \alpha u_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta Z_{t-i} + \epsilon_t \tag{15}$$

where $u_t = \beta' Z_t$ is the stationary [i.e., integrated of order zero] cointegration error of stock price.

So if the stock price is larger than its long run equilibrium $p_{t-1} > \gamma d_{t-1}$, then the stock price will go down or $\Delta p_t < 0$ when $\alpha < 0$. However if the stock price is smaller than its long run equilibrium $p_{t-1} < \gamma d_{t-1}$, then the stock price will go up or $\Delta p_t > 0$. In short, there is an error correcting mechanism.[3] Our claim is that the long run equilibrium γd_{t-1} may have a stochastic trend that is interpreted as a kind of bubble in terms of rational bubble concept. That error correcting mechanism may support a self fulfilling expectation on the future stock price with bubble: i.e., an invest sure the bubble is a part of sustainable long run (cointegration) equilibrium.

Then I suggest a candidate of SVAR model of (13) is defined as[4]

$$\Pi_0 Z_t = \sum_{i=1}^{p} \Pi_0 \Pi_i Z_{t-i} + e_t$$

or

$$Z_t^* = \sum_{i=1}^{p} \Pi_i^* Z_{t-i}^* + e_t \tag{16}$$

where the identification order following Sims (1980) is from the dividend to the stock price as in Wu (1997) with

$$\Pi_0 \equiv \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}, \quad |\Pi_0| \neq 0$$

as in (12) where $Z_t^* = \Pi_0 Z_t = (d_t, u_t)$, $\Pi_i^* = \Pi_0 \Pi_i \Pi_0^{-1}$ and $e_t = \Pi_0 \epsilon_t = \left(\delta_t, \xi_{\gamma t}\right)'$ with $\xi_{\gamma t} \equiv \xi_t - \gamma \delta_t$.

Then it is helpful if we write model (16) as

$$Z_t^* = \bar{\Pi}^* Z_{t-1}^* + \sum_{i=1}^{p-1} \Phi_i^* \Delta Z_{t-i}^* + e_t, \tag{17}$$

where $\Pi_i^* = \Pi_0 \Pi_i \Pi_0^{-1}$, $\Phi_i^* \equiv -\left(\Pi_{i+1}^* + \Pi_{i+2}^* + ... + \Pi_p^*\right)$ for $i = 1, 2, ..., p$, and $\bar{\Pi}^* \equiv$

[2] It is well known, we can not simply write a difference form of VAR model for ΔZ_t if there is a cointegration relation.

[3] See Zivot and Wang (2006) for the nice explanation of this mechanism.

[4] So our claim (16) is that the cointegration relation represents the structure of VAR model. However, for this SVAR structure to be valid for the standard dynamic analyses including variance decomposition and impulse response computation, the elements in error e_t should be orthogonal: i.e., the endogeneity of VAR model is fully captured in the matrix Π_0. So we will suggest a filtering through a projection to make this claim hold. See Section 4.1 for details.

$\sum_{i=1}^{p} \Pi_i^*$. Then we may verify that $\bar{\Pi}^*$ of equation (17) is arranged as

Proposition 2 *Suppose* Assumption 1 *holds. Then*

$$\bar{\Pi}^* = \begin{pmatrix} 1 & \alpha_1 \\ 0 & \beta\alpha' + 1 \end{pmatrix}.$$

The proofs of all propositions and theorems are in Appendix. Note, in (17), both the dependent and explanatory variables have the integrated of order one variables d_t and d_{t-1}. This obviously hinders to use a standard inference. Thus, we transform the model of Z_t^* into a VAR model of a purely stationary variable $W_t = (\Delta d_t, u_t)'$ using Proposition 1. More specifically, applying Kim and Park (2008), we may write the SVAR (16) as the VAR with the stationary variables or transformed error correction model [TECM henceforth] as:[5]

Theorem 3 *Suppose* Assumption 1 *holds. Then*[6]

$$W_t = \sum_{i=1}^{p} \Psi_i W_{t-i} + e_t \tag{18}$$

where

$$\Psi_i = \begin{pmatrix} \psi_{11i} & \psi_{12i} \\ \psi_{21i} & \psi_{22i} \end{pmatrix}; \; i = 1, 2...., p \text{ and } \psi_{11p} = \psi_{21p} = 0.$$

So if $\Phi = \alpha\beta$ and there is an ECM, then the representation (18) exists. Further note the zero coefficients in Ψ_p restricts that Δd_{t-p} does not appear in equation (32).[7]

Then suppose $\left| I_2 - \sum_{i=1}^{p-1} L^{p-1}\Psi_i \right| \neq 0$, then a vector moving average representation of

[5] See Appendix for the proof of our case.

[6] If the cointegration coefficient γ is estimated as $\hat{\gamma}$, then $\hat{u}_t = (-\gamma, 1)\, Z_t$ may replace u_t. However it does not affect to the asymptotic distributions of estimators or test statistics. See Theorem 4. Bohl (2003) just use a AR(p) process of $\hat{u}_t$ applying Enders-Siklos momentum threshold autoregressive model, to test the existence of periodically collapsing bubbles. In our SVAR structure, it may induce a misspecification error since the dividend changes are omitted in the explanatory variables. Especially, the lagged dividend changes Δd_{t-1} and Δd_{t-2} show the significant coefficient estimators in the US data application. See Table 1 in Section 5.

[7] This zero restriction should be considered to estimate the coefficient Ψ_i. If the redundant variable Δd_{t-p} is added, then it will decrease the estimation efficency even though the consistency of estimation holds. The system (18) is also stated in Campbell and Shiller (equation 5, 1987) without referring three facts: the VAR model (13) of levels, a necessary restriction $\Phi = \alpha\beta$ and zero restriction for the coefficient Ψ_p.

W_t in (18) is defined as:

$$\begin{aligned} W_t &= \left[I_2 - \sum_{i=1}^{p} L^{i-1}\Psi_i\right]^{-1} e_t \qquad (19) \\ &= \begin{pmatrix} 1-\sum_{i=1}^{p-1} L^{i-1}\psi_{11i} & -\sum_{i=1}^{p} L^{i-1}\psi_{12i} \\ -\sum_{i=1}^{p-1} L^{i-1}\psi_{21i} & 1-\sum_{i=1}^{p} L^{i-1}\psi_{22i} \end{pmatrix}^{-1} e_t \\ &= \frac{\begin{bmatrix} 1-\sum_{i=1}^{p} L^{i-1}\psi_{22i} & \sum_{i=1}^{p} L^{i-1}\psi_{12i} \\ \sum_{i=1}^{p-1} L^{i-1}\psi_{21i} & 1-\sum_{i=1}^{p-1} L^{i-1}\psi_{11i} \end{bmatrix}}{\left(1-\sum_{i=1}^{p-1} L^{i-1}\psi_{11i}\right)\left(1-\sum_{i=1}^{p} L^{i-1}\psi_{22i}\right)} e_t \\ &\equiv \sum_{i=0}^{\infty} \begin{pmatrix} \theta_{11i}L^{i-1} & \theta_{12i}L^{i-1} \\ \theta_{21i}L^{i-1} & \theta_{22i}L^{i-1} \end{pmatrix} \begin{pmatrix} \delta_t \\ \xi_{\gamma t} \end{pmatrix}. \end{aligned}$$

4 Estimation and test of SBT

Wu (1997) estimated the rational bubble directly assuming an orhtogonality condition as $E\eta_t\delta_t = 0$ for the Kalman filtering method. However I rather follow a direct approach: first deduce a candidate of SBT from the stock price via Beveridge-Nelson decomposition, second filter it through a projection to meet an orhtogonality condition for the SBT.

4.1 Beveridge-Nelson decomposition

Beveridge and Nelson (1981) show that the log of stock price p_τ which is integrated of order one, may be decomposed into two component: one is trend p_{Tt} =$\lim_{i\to\infty} E_t(p_\infty)$ and the other is cyclical part.[8] My claim is that the trend component p_{Tt} may again be decomposed into two part: one is due to the fundamental and the other is due to the bubble in a VAR model.

To show this, note the Beveridge-Nelson decomposition of dividend implies

$$d_\tau = d_{d0} + \theta_{11}(1)\sum_{i=1}^{\tau}\delta_i + \theta_{12}(1)\sum_{i=1}^{\tau}\xi_{\gamma i} + \eta_\tau - \eta_0$$

since $\Delta d_t = \sum_{i=0}^{\infty}\left(\theta_{11i}L^{i-1}\delta_t + \theta_{12i}L^{i-1}\xi_{\gamma t}\right)$ from (19) where $\xi_{\gamma j} \equiv \xi_j - \gamma\delta_j$, $\theta_{11}(1) = \sum_{i=1}^{\infty}\theta_{11i}$, $\theta_{12}(1) = \sum_{i=1}^{\infty}\theta_{12i}$, $\sum_{i=1}^{t} j^{1/2}|\theta_{1\ell j}| < \infty$, for $\ell = 1, 2$ and $\eta_\tau - \eta_0$ is a stationary process. Therefore we get

$$p_\tau = p_0 + \gamma\theta_{11}(1)\sum_{i=1}^{\tau}\delta_i + \gamma\theta_{12}(1)\sum_{i=1}^{\tau}\xi_{\gamma i} + \eta_\tau - \eta_0 \qquad (20)$$

[8] The Beveridge-Nelson decomposition has been subject to two different interpretations. One interpretation, by Watson (1986) and Morley, Nelson, and Zivot (2003), is that the longhorizon conditional forecast used to compute the Beveridge-Nelson trend corresponds to an estimate of the permanent component of an integrated series. A second interpretation, in Beveridge and Nelson (1981), is that the Beveridge-Nelson trend provides a definition of the permanent component of an integrated series.

since $p_t = \gamma d_t + u_t$ using the cointegration relation $u_t = \beta' Z_t$.

Assume (ξ_t, δ_t) is an i.i.d. normal process. Then note $\xi_t - \gamma\delta_t$ is independent with δ_t for all t if $E(\xi_t - \gamma\delta_t)\delta_t = 0$.[9] If $E(\xi_t - \gamma\delta_t)\delta_t \neq 0$, then define a $\bar{\gamma}$ such that $E(\xi_t - \bar{\gamma}\delta_t)\delta_t = 0$ where $\bar{\gamma} \equiv E\xi_t\delta_t / E\delta_t^2$. Thus we may define a conformable variable $\xi_{\bar{\gamma}t} = \xi_t - \bar{\gamma}\delta_t$ which is not correlated with δ_t.[10] Now above decomposition (20) is modified as:

$$p_\tau = p_0 + [\gamma\theta_{11}(1) + (\bar{\gamma} - \gamma)\gamma\theta_{12}(1)] \sum_{i=1}^{\tau} \delta_i + \gamma\theta_{12}(1) \sum_{i=1}^{\tau} \xi_{\bar{\gamma}i} + \eta_\tau - \eta_0 \tag{21}$$

since $\xi_{\gamma t} = \xi_{\bar{\gamma}t} + (\bar{\gamma} - \gamma)\delta_t$ from definition.

Three remarks are noteworthy for the new decomposition (21). First, in above decomposition, the part $\sum_{i=1}^{\tau} \xi_{\bar{\gamma}i}$ in (21) is an I(1) random walk process and independent with another I(1) random walk process of pure dividend shock $\sum_{i=1}^{\tau} \delta_i$. In this sense, we may regard the part $\gamma\theta_{12}(1)\sum_{i=1}^{\tau} \xi_{\bar{\gamma}i}$ as the SBT. Second, the SBT does not exist if $\theta_{12}(1) = 0$ or $\sum_{i=1}^{p} \psi_{12i} = 0$ since

$$\theta_{12}(1) = \frac{\sum_{i=1}^{p} \psi_{12i}}{\left(1 - \sum_{i=1}^{p-1} \psi_{11i}\right)\left(1 - \sum_{i=1}^{p} \psi_{22i}\right)} \tag{22}$$

from (19) if $\gamma \neq 0$. Third, the part $[\gamma\theta_{11}(1) + (\bar{\gamma} - \gamma)\gamma\theta_{12}(1)] \sum_{i=1}^{\tau} \delta_i$ signifies the stochastic trend of dividend shocks [henceforth SDT].

4.2 Estimation and testing

To estimate the SBT for p_τ, we use following steps:

1. Estimate Π_i as $\hat{\Pi}_i$ for all $i = 1, 2.., p$ from the VAR model (13) and get residual $\left(\hat{\delta}_j, \hat{\xi}_j\right)'; j = 1, 2, ..., n$.
2. Estimate $\bar{\gamma}$ as $\hat{\bar{\gamma}} \equiv \sum_{i=1}^{n} \hat{\xi}_i \hat{\delta}_i / \sum_{i=1}^{n} \hat{\delta}_i^2$ using a projection.
3. Compute the estimator of $\xi_{\bar{\gamma}j}$ as $\hat{\xi}_{\bar{\gamma}j} = \hat{\xi}_j - \hat{\bar{\gamma}}\hat{\delta}_j; j = 1, 2, ..., n$.
4. Estimate the cointegration coefficient γ as $\hat{\gamma}$ using Johansen (1991).
5. Estimate u_j as $\hat{u}_j = (-\hat{\gamma}, 1) Z_j$.
6. Run a regression (18) replacing u_j into $\hat{u}_j$ to get the estimators of ψ_{ijk} as $\hat{\psi}_{ijk}; i, j = 1, 2$ and $k = 1, 2, ..., p$.
7. Compute the estimator of $\theta_{12}(1)$ as

$$\hat{\theta}_{12}(1) = \sum_{i=1}^{p} \hat{\psi}_{12i} / \left(1 - \sum_{i=1}^{p-1} \hat{\psi}_{11i}\right)\left(1 - \sum_{i=1}^{p} \hat{\psi}_{22i}\right).$$

[9] This is what we assumed to justify the structure of SVAR (16).

[10] The variable $\xi_{\bar{\gamma}t}$ is the population projection error of the shock to the log transformed stock price ξ_t on the shock δ_t to the log transformed dividend.

8. Compute the SBT as $\hat{\gamma}\hat{\theta}_{12}(1)\sum_{i=1}^{\tau}\hat{\xi}_{\bar{\gamma}i}$; $\tau = 1, 2, ..., n$.

To estimate the SDT for p_τ, we follow following steps:

1. Compute the estimator of $\theta_{11}(1)$ as

$$\hat{\theta}_{11}(1) = \left(1 - \sum_{i=1}^{p}\hat{\psi}_{22i}\right) / \left(1 - \sum_{i=1}^{p-1}\hat{\psi}_{11i}\right)\left(1 - \sum_{i=1}^{p}\hat{\psi}_{22i}\right)$$

where

$$\theta_{11}(1) = \frac{1 - \sum_{i=1}^{p}\psi_{22i}}{\left(1 - \sum_{i=1}^{p-1}\psi_{11i}\right)\left(1 - \sum_{i=1}^{p}\psi_{22i}\right)}$$

from (19).

2. Estimate $\left[\hat{\gamma}\hat{\theta}_{11}(1) + \left(\hat{\bar{\gamma}} - \hat{\gamma}\right)\hat{\gamma}\hat{\theta}_{12}(1)\right]\sum_{i=1}^{\tau}\hat{\delta}_i$.

To test the existence of SBT for p_τ, the null hypothesis becomes $H_0 : \sum_{i=1}^{p}\psi_{12i} = 0$ since it implies $\theta_{12}(1) = 0$ from the definition (22).

1. Run a regression (18) replacing u_j into $\hat{u}_j$.
2. Conduct a t-test to check the null.

We may show that above estimated SBT and SDT are consistent for any give t. Further suggested t-test has the standard limit distribution. These properties are due to the super-consistency of cointegration coefficient $\hat{\gamma}$.

Theorem 4 *Suppose* Assumption 1 *holds. Then*

(a) $var\left(\sum_{i=1}^{p}\hat{\psi}_{12i}\right)^{-1/2} n^{1/2}\sum_{i=1}^{p}\hat{\psi}_{12i} \xrightarrow{d} N(0,1)$ *where* $var(\cdot)$ *denotes the asymptotic variance of the argument.*

(b) $\hat{\gamma}\hat{\theta}_{12}(1)\sum_{i=1}^{\tau}\hat{\xi}_{\bar{\gamma}i} \xrightarrow{p} \gamma\theta_{12}(1)\sum_{i=1}^{\tau}\xi_{\bar{\gamma}i}$ *for any given* t.

(c) $\left[\hat{\gamma}\hat{\theta}_{11}(1) + \left(\hat{\bar{\gamma}} - \hat{\gamma}\right)\hat{\gamma}\hat{\theta}_{12}(1)\right]\sum_{i=1}^{\tau}\hat{\delta}_i \xrightarrow{p} \left[\gamma\theta_{11}(1) + (\bar{\gamma} - \gamma)\gamma\theta_{12}(1)\right]\sum_{i=1}^{\tau}\delta_i$ *for any given* t.

5 Results for US stock prices

The data are identical to the monthly data used by Diba and Grossman (1988a) and many others, but extended to year 2010: real S&P Composite stock prices, P_t, and associated real dividends, D_t, available from Robert J. Shillers web-page (http://www.robertshiller.com). As a first preliminary check on the time-series properties of demeaned log-transformed prices and dividends,[11] we conduct Augmented Dickey-Fuller tests for unit roots. The resulting ADF-values for d_t and p_t are respectively, -1.7432 and -1.2485. Both statistics signify that

[11] A time trend term is not considered because of its insignificance.

we can not reject the null of unit root with 1% significance level. The estimation of VAR models for d_t and p_t shows that the order is 3 from Schwarz criterion. Following Fig. 1 plots the time series of estimated residuals of this VAR model. In there, the dividend equation residual series ($\hat{\delta}_i$) seems to be less volatile than the stock price equation residual series ($\hat{\xi}_i$).

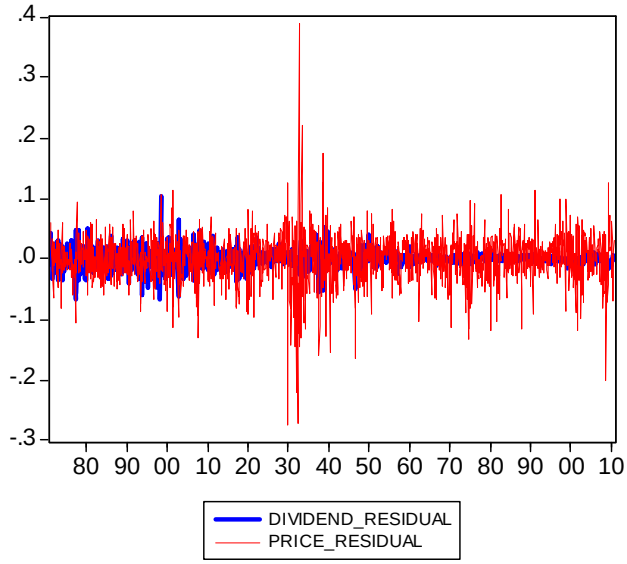

Figure 1. The dividend equation residual series (DIVIDEND_RESIDUAL $\hat{\delta}_i$) and the stock price equation residual series (PRICE_RESIDUAL $\hat{\xi}_i$).

Then a TECM in (18) is estimated as in Table 1. The F-test in (18) using estimated cointegration errors shows that the null $\mathrm{H}_0 : \sum_{i=2}^{5} \psi_{12i} = 0$ is rejected with 1% significance level. This supports that there may be a STB. Then Johansen cointegration test for d_t and p_t showed the null of no-cointegration may be rejected with 1% significance level since the trace and max-eigen test statistics are 38.3 and 37.1 respectively.

The cointegration relation is estimated as $p_t = 1.7746 d_t$. The residual based cointegration test also showed a similar result and the null of no-cointegration may be rejected with 1% significance level. Further we get $\hat{\theta}_{11}(1) = 2.2301$ and $\hat{\theta}_{12}(1) = 1.1302$ and $\widehat{\widehat{\gamma}} = 0.4079$ with the t-value=5.1463. For the unit root tests for the SBT, SDT and cyclical part, the null of unit root can not be rejected with 10% significance level while it can be rejected with the same level for the cyclical part.

Fig. 2 plots real stock prices and (scaled) real dividends over the period 1871.1-2010.9: Note d_t and p_t move together tightly except for the period mid 2000s supporting that there is a cointegration relation.

dependent variables	explanatory variables					Adjusted R^2	Durbin-Watson stat.
	Δd_{t-1}	Δd_{t-2}	$\hat{u}_{t-1}$	$\hat{u}_{t-2}$	$\hat{u}_{t-3}$		
Δd_{t-1}	0.3787	0.1729	-0.0306	0.0404	-0.0039	0.3254	2.0120
t-value	14.464	6.7269	-4.1320	3.3876	-0.5241		
$\hat{u}_{t-1}$	-0.2355	-0.2075	1.3492	-0.4422	0.0816	0.9824	2.0057
t-value	-2.5229	-2.2643	50.933	-10.394	3.0320		

Table 1: TECM estimated coeffcients

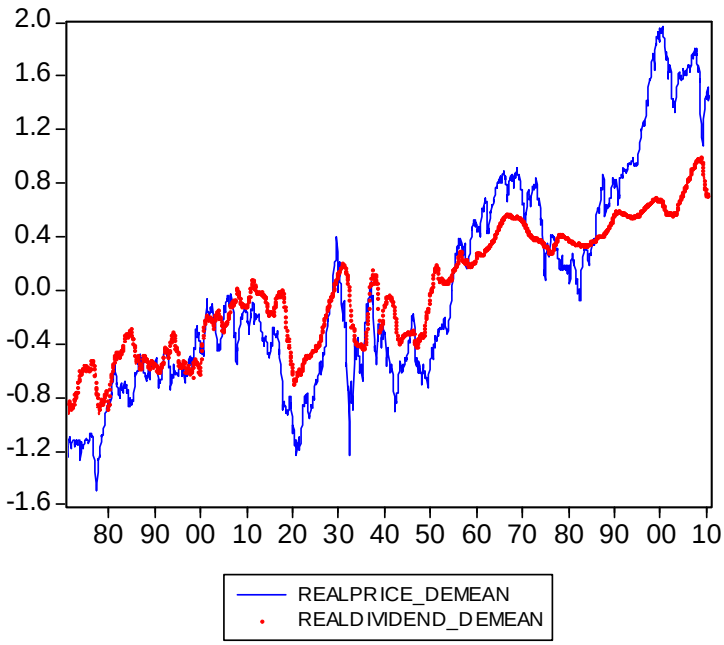

Figure 2. log transformed and demeaned real stock price (REALPRICE_DEMEAN) and real dividend (REALDIVIDEND_DEMEAN)

Following Fig. 3 shows that the stochastic trend of pure dividend seems to be relatively stable. However the SBT seems to be quite volatile. See the hikes of SBT during 1900, late 1920s (Great depression), 2000s (Global financial crisis). For these periods, the SBT was larger than the SDT and the fluctuation of stock price is rather explained by the SBT than the SDT.

Cointegration test results also confirmed these conjectures based upon graphs. Johansen cointegration test for p_t and SBT showed the null of no-cointegration may be rejected with 1% significance level since the trace and max-eigen test statistics are 54.2 and 51.2 respectively. The cointegration relation is estimated as $p_t = 0.3330d_t$. However Johansen cointegration test for p_t and SDT showed the null of no-cointegration can not be rejected

with 10% significance level since the trace and max-eigen test statistics are 5.56 and 5.16 respectively.

These results are coincided with the earlier findings of Shiller (1981), LeRoy and Porter (1981): i.e., stock prices are too volatile to be attributed to market fundamentals failure of simple present value model and there is an error correction mechanism where the stock price converges to its long run equilibrium which is dominated by the SBT.

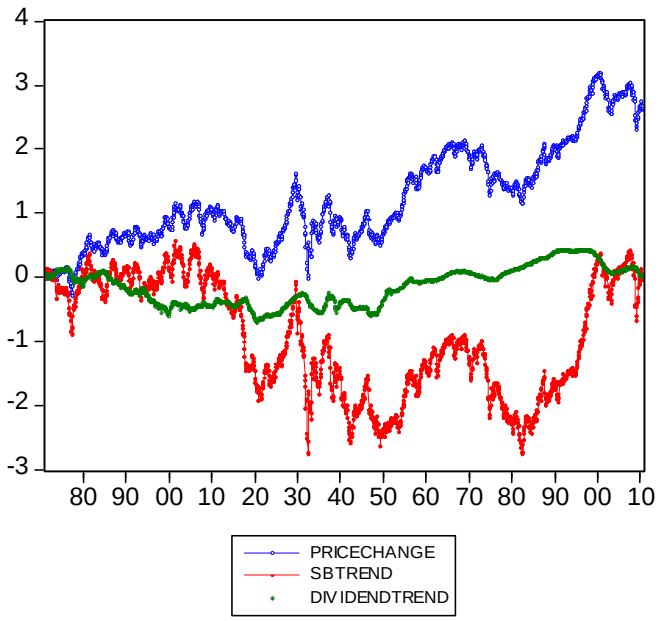

Figure 3. log transformed and demeaned real stock price change as $p_t - p_0$ (REAL-PRICE_DEMEAN), SBT (SBTREND) and SDT (DIVIDENDTREND)

6 Conclusion

The paper suggests there may be an I(1) stochastic bubble trend in the stock price even if the I(1) stock price and dividend are cointegrated that are usually confirmed in the empirical tests. For this, we show the long run equilibrium of stock price may be decomposed of fundamental and bubble stochastic trends; i.e., the sum of dividend innovations and the sum of innovations that are orthogonal with the dividend innovations, through the Beveridge Nelson decomposition and projection.

Two remained works are noteworthy. At first, as argued by Watson (1986), note the Beveridge-Nelson decomposition is a plausible candidate for modeling trend and cycle, but not the only possibility. So the comparing other approaches with the introduced SBT

computation deserves. At second, the fluctuation of SBT may be theoretically explained. For instance, the periodically collapsing bubbles in Evans (1991) is a possibility.

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Appendix : Mathematical Proofs

Proposition 2 First, we write from the definition

$$\begin{aligned} \bar{\Pi}^* &\equiv \Pi_0 \bar{\Pi} \Pi_0^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix} \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \\ &= \begin{pmatrix} \pi_{11} + \pi_{12}\gamma & \pi_{12} \\ -\gamma\pi_{11} + \pi_{21} - \gamma^2\pi_{12} + \pi_{22}\gamma & -\gamma\pi_{12} + \pi_{22} \end{pmatrix} \end{aligned} \tag{23}$$

using

$$\Pi_0^{-1} = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

for the first equality, where $\bar{\Pi} \equiv \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix}$.

We now represent the submatrices of $\bar{\Pi}$ of the third equality of (23) by using $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (-\gamma, 1)$, where $\Phi = \alpha\beta'$ under Assumption 1. For this, note that

$$\begin{aligned} \bar{\Pi} &\equiv \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \\ &= \alpha\beta' + I_2 \\ &= \begin{pmatrix} -\alpha_1\gamma + 1 & \alpha_1 \\ -\alpha_2\gamma & \alpha_2 + 1 \end{pmatrix} \end{aligned} \tag{24}$$

because the matrix Φ can be written as

$$\Phi = \alpha\beta' = \bar{\Pi} - I_2 \tag{25}$$

from the coefficient definition of (14) for the second equality.

Then the third equality of Equation (24) directly implies that

$$\pi_{11} = -\alpha_1\gamma + 1, \tag{26}$$
$$\pi_{12} = \alpha_1, \tag{27}$$
$$\pi_{21} = -\alpha_2\gamma, \tag{28}$$
$$\pi_{22} = \alpha_2 + 1. \tag{29}$$

Consequently, if we plug the submatrices (26) - (29) into the last term of Equation (23), then we get following:

(i) $\pi_{11} + \pi_{12}\gamma = -\alpha_1\gamma + 1 + \alpha_1\gamma = 1$,

(ii) $\pi_{12} = \alpha_1$,

(iii) $-\gamma\pi_{11} + \pi_{21} - \gamma^2\pi_{12} + \pi_{22}\gamma = \gamma^2\alpha_1 - \gamma - \alpha_2\gamma - \gamma^2\alpha_1 + \alpha_2\gamma + \gamma = 0$,

(iv) $-\gamma\pi_{12} + \pi_{22} = -\gamma\alpha_1 + \alpha_2 + 1 = \beta'\alpha + 1$.

The above results (i) - (iv), with the equality of (23), proves the claimed result as

$$\bar{\Pi}^* = \begin{pmatrix} 1 & \alpha_1 \\ 0 & \beta\alpha' + 1 \end{pmatrix}$$

from (23). ■

Theorem 3 Thus, we show how we may transform the model of Z_t^* into a VAR model of a purely stationary variable $W_t = (\Delta d_t, u_t)'$ using Proposition 2. For this purpose, using Proposition 2, we first rewrite Model (17) as

$$\begin{pmatrix} \Delta d_t \\ u_t \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \beta\alpha' + 1 \end{pmatrix} u_{t-1} + \textstyle\sum_{i=1}^{p-1} \Phi_i^* \Delta Z_{t-i}^* + e_t \tag{30}$$

since

$$\bar{\Pi}^* Z_{t-1}^* = \begin{pmatrix} d_{t-1} + \alpha_1 u_{t-1} \\ [\beta\alpha' + 1]\, u_{t-1} \end{pmatrix}.$$

The right hand side of (30) may be written as

$$\lambda u_{t-1} + \textstyle\sum_{i=1}^{p-1} \left[\phi_1^i \Delta d_{t-i} + \phi_2^i \left(u_{t-i} - u_{t-i-1}\right)\right] + e_t \tag{31}$$

defining $\Phi_i^* \equiv \left(\phi_1^i, \phi_2^i\right)$ conformably and

$$\lambda \equiv \begin{pmatrix} \alpha_1 \\ \beta\alpha' + 1 \end{pmatrix}.$$

Finally, the terms in (31) may be written as

$$\begin{aligned} & \left(\lambda+\phi_2^1\right) u_{t-1}+\sum_{i=1}^{p-1} \phi_1^i \Delta d_{t-i}+\sum_{i=2}^{p-1}\left(\phi_2^i-\phi_2^{i-1}\right) u_{t-i}-\phi_2^{p-1} u_{t-p}+e_t \\ = & \left(\phi_1^1, \lambda+\phi_2^1\right)\left(\begin{array}{c} \Delta d_{t-1} \\ u_{t-1} \end{array}\right)+ \\ & \sum_{i=2}^{p-1}\left(\phi_1^i, \phi_2^i-\phi_2^{i-1}\right)\left(\begin{array}{c} \Delta d_{t-i} \\ u_{t-i} \end{array}\right)+\left(0,-\phi_2^1\right)\left(\begin{array}{c} \Delta d_{t-p} \\ u_{t-p} \end{array}\right)+e_t \end{aligned}$$

or a p-th order VAR model of W_t as

$$W_t=\sum_{i=1}^p \Psi_i W_{t-i}+e_t, \tag{32}$$

where $\Psi_1=\left(\phi_1^1, \lambda+\phi_2^1\right)$, $\Psi_i=\left(\phi_1^i, \phi_2^i-\phi_2^{i-1}\right); i=2,3,...,p-1$ and $\Psi_p=\left(0,-\phi_2^1\right)$ conformably where $W_{t-i}=\left(\Delta d_{t-i}, u_{t-i}\right)'$. ■

Theorem 4 Since it may be mechanically generalized, the proof is conducted for a VAR (1) model as:

$$Z_t=\Pi Z_{t-1}+\epsilon_t \tag{33}$$

where $Z_t=\left(d_t, p_t\right)'$. Define $W_i=\left(\Delta d_i, u_i\right)'$, $\hat{W}_t=\left(\Delta d_i, \hat{u}_i\right)'$. Let a stacked form of a variable x_i be a $n \times 1$ vector $x \equiv\left(x_1, x_2, ..., x_n\right)'$ and $x_L \equiv\left(x_0, x_1, ..., x_{n-1}\right)'$.

Proof of (a): We follow next steps as:

(i) $n^{1/2}(\hat{\gamma}-\gamma) \xrightarrow{p} 0$ from Johansen (1991).

(ii) By definition, $\hat{u}_t-u_t=(\gamma-\hat{\gamma}) d_t$.

(iii-1) $n^{-1} d' \Delta d$, $n^{-1} d' e$, $n^{-1} d' u$, $n^{-2} d' d$, $n^{-1} W\prime W\prime$, $n^{-1/2} W_L' e$ are $O_p(1)$ applying Park and Phillips (1988, 1989).

(iii-2) $n^{-1} d' \hat{u}=n^{-1} d'(\hat{u}-u)+n^{-1} d' u$ is $O_p(1)$ from (iii-1) and

$$n^{-1} d'(\hat{u}-u)=n^{-1}(\gamma-\hat{\gamma}) n^{-2} d' d=O_p(1).$$

(iii-3) $n^{-1} \hat{W}' d=n^{-1}\left(\Delta d' d, \hat{u}' d\right)'$ is $O_p(1)$ from (iii-1) and (iii-2).

A TECM of (33) is written as:

$$W_t=\Psi W_{t-1}+e_t$$

or

$$\begin{aligned} \hat{W}_t & = \Psi \hat{W}_{t-1}+\Psi\left(W_{t-1}-\hat{W}_{t-1}\right)+\left(\hat{W}_t-W_t\right)+e_t \\ & = \Psi \hat{W}_{t-1}+\Psi\left(0, u_{t-1}-\hat{u}_{t-1}\right)'+\left(0, u_t-\hat{u}_t\right)'+e_t \end{aligned} \tag{34}$$

after an arrangement and rewriting (34) as:

$$\begin{pmatrix} \Delta d_t \\ \hat{u}_t \end{pmatrix} = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} \hat{W}_{t-1} + \begin{pmatrix} \psi_{12} \\ \psi_{22} \end{pmatrix} (u_{t-1} - \hat{u}_{t-1}) + \begin{pmatrix} 0 \\ u_t - \hat{u}_t \end{pmatrix} + e_t. \tag{35}$$

where

$$\Psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}.$$

Define an OLS estimator of the first row $\psi_1 \equiv \begin{pmatrix} \psi_{11} & \psi_{12} \end{pmatrix}$ in the equation of (35) as $\hat{\psi}_1 \equiv (W_L' W_L)^{-1} W_L' \Delta d$.

(vi) Slutsky's theorem implies

$$n^{1/2}\left(\hat{\psi}_1 - \psi_1\right) \xrightarrow{d} N\left[0,\ \text{plim}\left(n^{-1} W_L' W_L\right)^{-1}\right] \tag{36}$$

since s standard central limit theorem holds

$$B \xrightarrow{d} N\left[0,\ \text{plim}\left(n^{-1} W_L' W_L\right)^{-1}\right]$$

where $B \equiv \left(n^{-1} W_L' W_L\right)^{-1} n^{-1/2} W_L' e$, and

$$n^{1/2}\left(\hat{\psi}_1 - \psi_1\right) - B \xrightarrow{p} 0$$

using following Facts 1, 2 and 3:

Fact 1:

$$\begin{aligned} n^{1/2}\left(\hat{\psi}_1 - \psi_1\right) &= \left(n^{-1}\hat{W}_L'\hat{W}_L\right)^{-1} n^{-1/2}\hat{W}_L' \left[\psi_{22}\left(u_L - \hat{u}_L\right) + (u - \hat{u}) + e\right] \\ &\equiv \hat{A} + \hat{B} \end{aligned}$$

from definition of $\hat{\psi}_1$ and the decomposition (35) where

$$\begin{aligned} \hat{A} &\equiv \left(n^{-1}\hat{W}_L'\hat{W}_L\right)^{-1} n^{-1/2}\hat{W}_L' \left[\psi_{22}\left(u_L - \hat{u}_L\right) + (u - \hat{u})\right], \\ \hat{B} &\equiv \left(n^{-1}\hat{W}_L'\hat{W}_L\right)^{-1} n^{-1/2}\hat{W}_L' e. \end{aligned}$$

Fact 2. $\hat{A} \xrightarrow{p} 0$:

$$n^{-1/2}\hat{W}_L'\left[\psi_{22}\left(u_L-\hat{u}_L\right)+\left(u-\hat{u}\right)\right] = \psi_{22}\left(\hat{\gamma}-\gamma\right)n^{-1/2}\hat{W}_L'\left(u_L-\hat{u}_L\right)+ n^{-1/2}\hat{W}_L'\left(u-\hat{u}\right)\xrightarrow{p}0 \tag{37}$$

and $n^{-1}\hat{W}_L'\hat{W}_L$ is $O_p(1)$ because $n^{-1}W'W$ is $O_p(1)$ and

$$\begin{aligned} &n^{-1}\hat{W}_L'\hat{W}_L-n^{-1}W_L'W_L= \\ &n^{-1}\left(\hat{W}_L-W_L\right)'\hat{W}_L+n^{-1}W_L'\left(\hat{W}_L-W_L\right)\xrightarrow{p}0 \end{aligned} \tag{38}$$

since

$$n^{-1}\left(\hat{W}_L-W_L\right)'\hat{W}_L=\left(\gamma-\hat{\gamma}\right)n^{-1}\begin{pmatrix} 0 & 0 \\ d_L'\Delta d_L & d_L'\hat{u}_L \end{pmatrix}\xrightarrow{p}0$$

and

$$n^{-1}W_L'\left(\hat{W}_L-W_L\right)=\left(\gamma-\hat{\gamma}\right)n^{-1}\begin{pmatrix} 0 & d_L'\Delta d_L \\ 0 & d_L'u_L \end{pmatrix}\xrightarrow{p}0$$

where $n^{-1}d_L'\Delta d_L$, $n^{-1}d_L'\hat{u}_L$, $n^{-1}d_L'u_L$ are $O_p(1)$ from (iii-1), (iii-2) and (iii-3).

Fact 3. $\hat{B}-B\xrightarrow{p}0$:

$$\begin{aligned} \hat{B}-B &= \left[\left(n^{-1}\hat{W}_L'\hat{W}_L\right)^{-1}-\left(n^{-1}W_L'W_L\right)^{-1}\right]n^{-1/2}\hat{W}_L'e \\ &\quad+\left(n^{-1}W_L'W_L\right)^{-1}n^{-1/2}\left(\hat{W}_L-W_L\right)'e\xrightarrow{p}0 \end{aligned}$$

from (38) and since

$$n^{-1/2}\left(\hat{W}_L-W_L\right)'e=n^{1/2}\left(\gamma-\hat{\gamma}\right)n^{-1}\begin{pmatrix} 0 \\ d_L\prime e \end{pmatrix}\xrightarrow{p}0. \tag{39}$$

from (i) and $n^{-1}d'e$ and $n^{-1/2}W_L'e$ are $O_p(1)$.

So the result (36) obviously implies the claimed result (a). ■

Proof of (b) and (c) : We use following facts:

(v) Obviously $\hat{\psi}_{ij}\xrightarrow{p}\psi_{ij}$ for i,j=1,2 holds from (vi). Thus $\hat{\theta}_{1\ell}(1)\xrightarrow{p}\theta_{1\ell}(1)$ for $\ell=1,2$ since $\theta_{1\ell}(1)$ is a continuous function of ψ_{ij} from (22).

(vi) By definition, note

$$\begin{aligned}\left(\hat{\delta},\hat{\xi}\right) &\equiv \hat{\epsilon}=Z-Z_L\hat{\Pi}'=\epsilon+P_Z\epsilon \\ &\equiv (\delta+P_Z\delta,\xi+P_Z\xi)\end{aligned}$$

where $P_Z=Z_L\left(Z_L'Z_L\right)^{-1}Z_L'$. The note

$$\hat{\bar{\gamma}}\xrightarrow{p}\bar{\gamma}\equiv E\xi_t\delta_t/E\delta_t^2$$

from following facts 1 and 2.

Fact 1: note

$$\begin{aligned}n^{-1}\hat{\delta}'\hat{\xi}-n^{-1}\delta'\xi &= 3n^{-1}\delta'P_Z\xi \\ &= 3n^{-1}\delta'Z_L\left(n^{-2}Z_L'Z_L\right)^{-1}n^{-2}Z_L'\xi\xrightarrow{p}0\end{aligned} \tag{40}$$

and

$$\begin{aligned}n^{-1}\hat{\delta}'\hat{\delta}-n^{-1}\delta'\delta &= 3n^{-1}\delta'P_Z\delta \\ &= 3n^{-1}\delta'Z_L\left(n^{-2}Z_L'Z_L\right)^{-1}n^{-2}Z_L'\delta\xrightarrow{p}0\end{aligned}$$

since $n^{-2}Z_L'Z_L$ is $O_p(1)$, $n^{-2}Z_L'\xi$ and $n^{-2}Z_L'\delta$ are $o_p(1)$.

Fact 2: $n^{-1}\delta\prime\xi\xrightarrow{p}E\xi_t\delta_t$ and $n^{-1}\delta'\delta\xrightarrow{p}E\delta_t^2$ from the law of large numbers using that (ξ_t,δ_t) is an i.i.d. sequence.

(vii) Note

$$\begin{aligned}\hat{\epsilon}-\epsilon &\equiv Z-Z_L\hat{\Pi}'-\epsilon \\ &= Z_L\left(\Pi-\hat{\Pi}\right)\prime\xrightarrow{p}0\end{aligned} \tag{41}$$

since $\hat{\Pi}-\Pi\xrightarrow{p}0$ from Park and Phillips (1988,1989) for a given Z_L.

Thus

$$\hat{\xi}_{\bar{\gamma}i}=\hat{\xi}_i-\hat{\bar{\gamma}}\hat{\delta}_i\xrightarrow{p}\hat{\xi}_{\bar{\gamma}i}=\xi_i-\bar{\gamma}\delta_i$$

using $\hat{\xi}_i-\xi_i\xrightarrow{p}0$, $\hat{\delta}_i-\delta_i\xrightarrow{p}0$ and $\hat{\bar{\gamma}}\xrightarrow{p}\bar{\gamma}$.

Consequently, we get $\sum_{i=1}^{\tau}\hat{\xi}_{\bar{\gamma}i}\xrightarrow{p}\sum_{i=1}^{\tau}\xi_{\bar{\gamma}i}$ and $\sum_{i=1}^{\tau}\hat{\delta}_i\xrightarrow{p}\sum_{i=1}^{\tau}\delta_i$ for any given $\tau=1,2,...,n$. ■