Markov Chain Approach for Bond Portfolio Selection

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Abstract

Mean-variance framework has been established as a standard portfolio management scheme since Markowitz's (1952) seminal work, however, it imposes some difficulties in practical implementation. The most difficult part is to forecast the future return and risk of individual investment assets. In this paper we propose a new approach modeling the ex-post optimal portfolio weights as a Markov process. We pay attention to the fact that the ex-post optimal investment is not a diversified investment but an all-or-nothing investment and we can classify individual assets as "invested" and "not-invested" each period. We model that the evolution of two states, "invested" and "not-invested", for each asset follows a Markov process. By modeling the portfolio weights directly, we can eliminate the uncertainty engaged with forecasting the future return and risk of assets. We apply the new approach of Markov process into bond portfolio selection with U.S and German treasuries. The cumulative return by applying the new approach proves to be superior to those by some simple traditional methods based on mean-variance framework.

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1 Introduction

Mean-variance framework has been established as a standard portfolio management scheme since Markowitz's (1952) seminal work, and we have observed its many extensions and modifications since then. The mean-variance framework is basically implemented through two stages: estimating the mean and variance-covariance of investment assets' return rates for a given holding period, and then calculating portfolio weights in order to maximize expected return given the investor's risk tolerance level. In spite of its simplicity and analytical tractability, however, it imposes some difficulties in practical implementation. The most difficult part is to forecast the future return and risk of individual investment assets, which has been most studied topics but has yielded few satisfactory results yet.

In this paper we propose a new approach of modeling the ex-post optimal portfolio weights as a Markov process. We pay attention to the fact that the ex-post optimal investment is not a diversified investment but an all-or-nothing investment and we can classify individual assets as "invested" and "not-invested" each period. We model that the evolution of two states, "invested" and "not-invested", for each asset follows a Markov process. By modeling the portfolio weights directly, we can eliminate the uncertainty engaged with forecasting the future return and risk of assets. We apply the new approach into bond portfolio selection with U.S. and German treasuries. The cumulative return by applying the new approach proves to be superior to those by some simple traditional methods based on mean-variance framework.

The remaining part of this paper is organized as follows: Next section introduces four traditional methods of bond portfolio selection for a performance comparison with the new approach: equal weighting, random walk, historical mean and AR(1) approaches. Section 3 discusses the new approach using a Markov chain, and its performance comparisons with the traditional ones are provided in section 4. Section 5 provides summary and concluding remarks.

2 Traditional Methods for Portfolio Selection

2.1 Markowitz's Mean-Variance Paradigm[1]

The mean-variance framework of Markowitz (1952) is the most popular paradigm for investors' optimal portfolio choice. Suppose an investor can invest in N risky assets and a risk-free asset. The N risky assets have a random return vector, $R = [R_1 \cdots R_N]'$, and the risk-free asset has known return R_f. The excess return on a risky asset is defined as the difference in returns between the risky and the risk-free assets. Its mean vector and covariance matrix are denoted as $\mu = [\mu_1 \cdots \mu_N]'$ and $\Sigma = [\sigma_{ij}]_{i,j=1,\cdots,N}$, respectively.

In this mean-variance paradigm, investors' problem is how to choose portfolio weights, $x = [x_1 \cdots x_N]'$, so as to minimize the variance of a portfolio given its expected return as below:

$$\begin{aligned} \text{Minimize } var[R_p] &= x'\Sigma x \text{ subject to } E[R_p] = x'\mu \equiv \overline{\mu}, \\ \text{where } R_p &= x'(R - R_f\mathbf{1}). \end{aligned} \tag{1}$$

The optimal portfolio weight x^*, which is the solution to the problem, is determined as follows:

$$x^* = (\overline{\mu}/\mu'\Sigma\mu)\Sigma^{-1}\mu. \tag{2}$$

Thus, investors allocate x^* of their wealth into risky assets and assign the rest into the risk-free asset.

As evident in the equation (2), we need the information on the mean vector and covariance matrix for excess returns in order to determine optimal portfolio weights. Since the information is not known in advance, it is indispensable to estimate the mean vector and covariance matrix in Markowitz's paradigm.

Previous research reports that it is extremely difficult to estimate expected returns[2] and that the optimization process is very sensitive to expected re-

[1] The exposition of Markowitz's mean-variance framework generally follows Brandt (2004).

[2] In this paper, we use 'mean' and 'expected return' interchangeably.

turns.[3] On the other hand, existing studies show that return variances and covariances are easier to estimate[4] and, thus, less problematic in the optimization process. In this vein, Chopra and Ziemba (1993) even show that errors in expected returns are about eleven times as important as errors in variance. Thus, in this paper, we apply several alternative methods to estimate expected returns whereas we compute variances and covariances with historical data in a usual way.

2.2 Alternative Ways to Estimate Expected Returns under Markowitz's Mean-Variance Paradigm

In this paper, we consider three alternative ways to estimate expected returns: (i) historical mean (HM), (ii) AR(1) and (iii) random walk (RW). Denote h as an investor's holding period. First, the expected return for next holding period is estimated as the simple average of historical returns for the past periods. Second, the expected return for next period is computed as a linear projection of a historical return realized h periods earlier. Lastly, the expected return for next holding period is assumed to be the same as the return for this period. Mathematically, these three approaches can be expressed as follows:

(i) Historical Mean: $\widehat{\mu}_t = (1/L)\sum_{i=0}^{L-1} r_{t-h-i}$

(ii) AR(1): $\widehat{\mu}_t = \alpha + \beta r_{t-h} + \varepsilon_t$

(iii) Random Walk: $\widehat{\mu}_t = r_{t-h}$

2.3 Equally-weighted Allocation Method

Another asset-allocation method we consider is an equally-weighted (EW) allocation strategy. Due to errors either in the estimation of parameters or in modeling, sophisticated asset-allocation methods are not guaranteed to outperform a simple method such as an equally-weighted allocation method. Indeed, DeMiguel, Garlappi, and Uppal (2005) compare the out-of-sample performance of an equally-weighted asset-allocation method to several static and dynamic

[3] Michaud(1989), Best and Grauer (1991)

[4] Nelson (1992), Nelson and Foster (1995)

models of optimal asset-allocation, and find that the equally-weighted asset-allocation strategy typically has a better out-of-sample outcome than optimal asset-allocation strategies.

3 Markov-Chain Asset-Allocation Approach

3.1 Markov Chains[5]

Suppose S_t be a random variable which can take only an integer value from 1 to M. Let's assume that the probability that the random variable takes a particular value of j depends only on the most recent realized value:

$$\text{Prob}\{S_t = j | S_{t-1} = i, S_{t-2} = k, \cdots\} = \text{Prob}\{S_t = j | S_{t-1} = i\} = p_{ij}. \quad (3)$$

This process is called an M-state Markov chain with transition probabilities $\{p_{ij}\}_{i,j=1,\cdots,M}$. The transition probability p_{ij} denotes the probability that state j will be the next state when state i is the most recent state. The transition probabilities are often collected in a matrix $\mathbf{P}$, usually described as a transition matrix:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1M} \\ p_{21} & p_{22} & \cdots & p_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ p_{M1} & p_{M2} & \cdots & p_{MM} \end{bmatrix}.$$

Note that each row should be summed to one, because we assume that next state should be one of the M states.

3.2 Markov-Chain Asset-Allocation Approach

We develop a new method in asset allocation using Markov chain approach. In our Markov-chain asset-allocation approach, we directly compute optimal

[5]For more details on the Markov Chain and their applications, refer to the chapter 22 of Hamilton (1994).

portfolio weights. Thus, our approach contrasts with Markowitz's mean-variance approach, where mean vector and covariance matrix for excess returns should be estimated before optimal portfolio weights are determined.

When some restrictions are imposed on portfolio weight x itself, for example, $x \in [u, v]$, the ex-post optimal portfolio is simply the all-or-nothing portfolio; that is, if the realized return of asset i was maximum among $(N+1)$ assets for a certain period, then the ex-post optimal portfolio is:

$$\begin{aligned} x_j^* &= u_j, \qquad j \neq i \\ x_i^* &= 1 - \sum_{j \neq i} u_j. \end{aligned} \tag{4}$$

Take an example of no-short-sale restriction. Then the resulting ex-post optimal portfolio become to invest all wealth into the asset with maximum realized return rate.[6]

At each period up to time $t-h$ we classify assets into the "invested" and "not-invested" assets where the invested asset is defined as the asset with maximum realized return rate at that time. Therefore, the "invested" and "not-invested" asset classes constitute two states and each asset belongs to a state at a time. Next, we model the evolution of the state for an asset to follow a Markov process. Specifically, denote b as the frequency for the Markov process to repeat and π_{ij} as the transition probability of asset j being at the state of "invested" at time t provided that asset i was at the state of "invested" at time $t-b$. Then the probability vector of being at the state of "invested" at time t, q_t, is expressed as:

$$\begin{aligned} q_t &= q_{t-b}\Pi, \\ q_t &= q_{t-h}\Pi^{h/b}, \end{aligned} \tag{5}$$

where π_{ij} is the $(i,j)th$ element of matrix Π.

[6] Without any restriction on portfolio weight x, we cannot set any specific ex-post optimal weight, and the optimal strategy is simply to invest into the realized "invested" asset as much as we can.

A natural estimate of π_{ij} is the ratio of the number that asset j is at the state of "invested" at time t when asset i was at the state of "invested" at time $t-b$ to the number of asset j being at the state of "invested". Once the matrix Π is estimated, the probability vector of being at the state of "invested" at time t, q_t, is readily obtained using the above relation.

We have two possibilities of forming ex-ante optimal portfolio weight using the probability vector q_t. The first one is simply to use q_t as our optimal portfolio weight. The second way is to form all-or-nothing strategy mimicking ex-post optimal portfolio weight by investing all available wealth into one asset with the highest probability of being "invested". It will be determined empirically which strategy is better. It will also be an empirical issue what value for (h/b) performs best.

4 Comparison of Performances under Alternative Asset-Allocation Approaches

In this section, using the data on U.S. and German government bond yields, we compare performances between the Markov chain (MC) approach and traditional ones. For the U.S. government bond yields, we use the data of 1 through 5 year zero-coupon bond prices[7] provided by Cochrane and Piazzesi (2005).[8] The sample period spans January 1968 through December 2003. For the German government bond yields, we collect the data of 1 through 5 year bond yields through the Deutsche Bundesbank.[9] The sample period is from January 1974 to December 2004. We assume that an investor holds a bond for a year ($h = 12$) and that there is no rebalancing during the period. Thus,

[7] Data of zero coupon bond prices is extensively used in the literature. For example, Fama and Bliss (1987) used the data to examine the information contents in forward rates. Campbell and Shiller (1991) also used the data to test the expectations theory of the term structure.

[8] The Cochrane-Piazzesi data is available at 'http://gsbwww.uchicago.edu/fac/monika.piazzesi/research/cp/bondprice.dat'.

[9] The data is available at 'http://www.bundesbank.de/statistik/statistik_zeitreihen.en.php?func=list&tr=www_s300_it02a'.

a bond with maturity of 1 year is regarded as a risk-free asset. For other bonds with maturity longer than 1 year, we use their annual returns when we compare performances between alternative approaches.

Figure 1 shows time trend in the risk-free rate and annual excess returns for bonds with maturity of 2 through 5 years. Excess returns for bonds with maturity of 2 through 5 years are defined as the difference in returns between those bonds and risk-free bond. From both Panel A (U.S. government bonds) and Panel B (German government bonds), we observe that excess returns are much more volatile than the risk-free rate. At the same time, all of the excess returns for bonds with maturity of 2 through 5 years seem to move together. This is more evident when we examine the time trend in Sharpe ratios for those bonds in Figure 2. The Sharpe ratios for those bonds are almost indistinguishable! Thus, we conclude that all the risky bonds share similar properties in terms of risk and return and use only one of those bonds, i.e. bond with maturity of 5 years in our analysis. This will simplify our analysis because the number of parameters we need to estimate will be significantly reduced.

As we discussed in section 2, we consider four traditional approaches. For some of the traditional approaches, we should determine how much we go back in the past in order to estimate either expected returns or covariance matrix. For historical mean and AR(1) approaches, we use historical returns for the last 5 years to estimate expected returns at a certain point of time. For historical mean, AR(1), and random walk approaches, we estimate covariance matrix with historical returns for the last 5 and 3 years of U.S. and German data, respectively. Then, we compute the optimal portfolio weights using the estimated expected returns and covariance matrix. We impose no-short-sale restriction on portfolio weights and assume the short-fall risk of 5% as the investor's risk tolerance level.

For the Markov-chain asset-allocation approach, we recursively estimate the transition probabilities between the risk-free asset and bond with maturity of 5 years by using all available data at each time. Then, all the money is invested into an asset, which is expected to have the higher return during next holding period. Figure 3 plots the estimated transition probabilities. For U.S. government bonds, the transition probability of risk-free asset being the "invested" is estimated above .5 all the time, meaning that if risk-free asset is realized as the "invested" at time $t-h$, then the optimal portfolio is to invest all available wealth into risk-free asset. However, the transition probability of risky asset being the "invested" is estimated below .5 roughly up to 1994 and above .5 thereafter, which designate the optimal portfolio as risk-free asset during the first period and as risky asset during the remaining period when the risky asset is realized as the "invested" at time $t-h$. We have different optimal portfolio selection rules for German government bonds. Since the estimated transition probability of risky asset being the "invested" is above .5 all the time, we would choose risky asset as the "invested" when the risky asset is realized as the "invested" at time $t-h$. However, if risk-free asset is realized as the "invested" at time $t-h$, we would invest into risk-free asset for the first several years and then into risky asset since then.

Figure 4 exhibits cumulative returns under Markov chain approach. Since we could start our investment in each month, we have twelve series of cumulative returns during the sample period. Specifically, we report the value of investment at a certain point of time as a way of expressing cumulative returns till that time. The value when an investor starts her investment is regarded as one. In Panel A where we discuss the case for U.S. government bonds, we assume that we started our investment in 1974. In this case, on average, the value of investment under Markov chain approach in 2003 is 10.78 (i.e. cumulative returns of 978%). In Panel B, we assume that we started investment in German government bonds in 1983 and find that the average value of investment under Markov chain approach in 2004 is 4.99 (i.e. cumulative returns of 399%).

From Figure 5.1 through Figure 8.2, we report cumulative returns under four traditional approaches and difference in cumulative returns between Markov chain approach and traditional approaches. In Figure 5.1 and Figure 5.2, equally-weighted asset-allocation method is considered for U.S. and German government bonds, respectively. For U.S. government bonds, the value of investment at the end of investment period is about 10 (cumulative returns of about 900%) as shown in Panel A of Figure 5.1. In Panel A of Figure 5.2, we observe that the corresponding investment value for German government bonds under equally-weighted asset-allocation method is about 4 (cumulative returns of about 300%) in 2004. Panel B of Figure 5.1 and Figure 5.2 shows that the differences in cumulative returns between Markov chain approach and equally-weighted asset-allocation method are positive in the most cases, which implies that Markov chain approach outperforms equally-weighted asset-allocation method in the sample. The other figures (historical mean approach in Figure 6.1 and Figure 6.2, random walk approach in Figure 7.1 and Figure 7.2, and AR(1) approach in Figure 8.1 and 8.2) show similar patterns as Figure 5.1 and Figure 5.2: For both U.S. and German government bonds, Markov chain approach outperforms traditional approaches during our sample period.

Table 1 summarizes comparison of performances under five alternative asset allocation approaches. Table 1 reports the value of investment at the end of investment period under Markov chain approach and the difference in the value between the Markov chain and each alternative approach. Again, we confirm that the differences in the mean value at the end of investment period between Markov chain and each alternative approach is positive and that Markov chain approach outperforms alternative approaches in almost every month at the end of investment period.[10]

These empirical results are based on the all-or-nothing strategy by investing all available wealth into one asset with the highest probability of being the "invested". We also tried the other possibility of using qt as our optimal portfolio weight, but the performance of Markov chain method was inferior to the case of all-or-nothing strategy. Since we have freedom in choosing the parameter of the repetition period of Markov chain b, we tried several bs, for example, $b = 1, 3, 6, 12$. The results were the same, so we report the results only with $b = 1$ for brevity. The reason for the same results is that the optimal portfolio would not be changed under the all-or-nothing strategy if different

[10] Recall that we have twelve series of cumulative returns during the sample period because we could start our investment in each month.

values for b do not change the estimates of the transition probabilities from below .5 to above .5 or from above .5 to below .5.

We mentioned above that we chose only bond with 5 year maturity as a risky asset. We also changed the risky asset with bonds of other maturities, and the results are also reported in Table 1. Performances become better under MC approach as bonds with shorter maturity are chosen as an investable asset for U.S. government bonds, while we obtain the reverse result for German government bonds. The numbers of months when MC approach performs worse than others are only up to 2 against EW method and 0 or 1 against other methods for U.S. treasuries. For German treasuries, there is none of the case.

5 Summary and Concluding Remarks

In this paper, we introduce Markov chain approach for investors' optimal portfolio choice. In the approach, we estimate the transition probabilities and directly decide optimal portfolio weights. Using the historical data on the U.S. and German government bond yields, we showed that our approach has outperformed traditional ones where expected returns are computed by historical mean, AR(1) or random walk. Our approach also performed better than equally-weighted allocation method. Thus, we suggest that our approach could be seriously considered as an alternative asset allocation strategy.

We have discussed only the basic idea of Markov chain approach in section 3. However, we could generalize the approach, particularly in terms of controlling the portfolio risk. There may be several ways to control the portfolio risk under the Markov chain approach; here we suggest a simple way of controlling the risk by changing the restriction on portfolio weight, $x \in [u, v]$. Under the mean-variance paradigm, the portfolio risk is controlled by changing the portfolio weights given the estimated mean and variance of asset returns. In contrast, we cannot change the portfolio weights given the restriction range for portfolio weights. However, observing that the restriction range for portfolio weights affects the portfolio risk, we may control the risk level by changing the restriction range for portfolio weights. By widening the range we probably end up with the increased portfolio risk; narrowing the range would reduce the portfolio risk. If we employ a parametric assumption for asset returns, we can obtain an analytic expression for the portfolio's return-risk profile. If we do not use a parametric modeling, we may use bootstrapping technique to estimate the return-risk profile numerically. To provide an illustrative effect of changing the restriction on portfolio weight, $x \in [u, v]$, we plot cumulative returns under MC approach with $u = .1, .2, .3$ in Figures 9.1 and 9.2, and also present the summary statistics in Table 2. As illustrated in Figures and the Table, as we restrict portfolio range tighter, the cumulative return under MC approach averaged over twelve months becomes lower with narrower dispersion among twelve series.

There is a caveat in our study. We couldn't find that our Markov chain approach outperforms other traditional approaches when money is invested into assets proportionately as if transition probabilities represent optimal portfolio weights at each state. This might be a significant drawback in terms of portfolio diversification, because our approach suggests that we need to invest our money only in one asset, which is expected to have the higher returns during next period. Thus, how to reconcile our Markov chain approach with the benefit of portfolio diversification should be the next task in our study.

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Table 1. The Value of Investment at the End of Investment Period under Alternative Asset Allocation Approaches: Investment in the U.S. and German Treasuries

Panel A: U.S. Government Bonds

Mat	MC	EW	HM	RW	AR
5	10.78	0.85 2	1.58 0	1.15 0	1.70 1
4	10.99	1.06 2	1.80 0	1.37 1	1.92 0
3	11.13	1.20 2	1.94 0	1.51 0	2.05 1
2	11.30	1.37 1	2.11 0	1.68 1	2.22 1

Panel B: German Government Bonds

Mat	MC	EW	HM	RW	AR
5	4.99	0.89 0	1.46 0	1.00 0	1.38 0
4	4.87	0.77 0	1.34 0	0.89 0	1.26 0
3	4.83	0.73 0	1.30 0	0.84 0	1.22 0
2	4.80	0.71 0	1.28 0	0.82 0	1.20 0

Note: “Mat” denotes the bond maturity included as investable assets together with risk-free bond with the maturity of one year. “MC” represents the mean value of investment at the end of investment period under MC approach. The mean value at the end of investment period is the mean of twelve values which represent the value of each month during the last year of investment period, respectively. The value when an investor starts her investment is regarded as one. The number in the first row in each cell for “EW”, “HM”, “RW”, and “AR” is the difference in the mean value between MC and the corresponding method. The number in the second row in the cell means the number of negative differences in cumulative returns among twelve months.

Table 2. Summary Statistics of Cumulative Returns under Markov Chain Approach with Various Portfolio Restrictions

Panel A: U.S. Government Bonds

u	Mean	s.d.	[Min, Max]
0.0	10.78	0.91	[9.69, 12.77]
0.1	10.48	0.76	[9.54, 12.10]
0.2	10.18	0.63	[9.33, 11.44]
0.3	9.87	0.52	[9.11, 10.80]

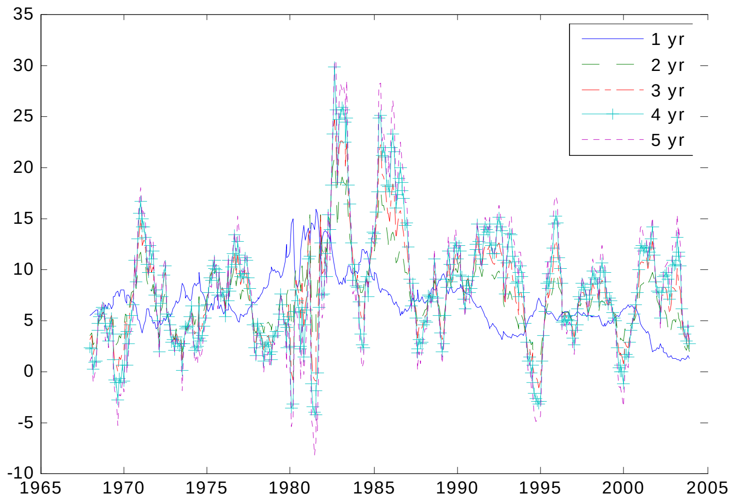
Panel B: German Government Bonds

u	Mean	s.d.	[Min, Max]
0.0	4.99	0.53	[3.98, 5.74]
0.1	4.76	0.42	[3.94, 5.33]
0.2	4.54	0.32	[3.89, 4.95]
0.3	4.33	0.23	[3.84, 4.60]

Note: Portfolio restrictions are imposed as $x \in [u, v]$.

Figure 1. Time Trend in the Risk-free Rate and Annual Excess Returns for Bonds with Maturity of 2 through 5 Years

Panel A: U.S. Government Bonds



Panel B: German Government Bonds

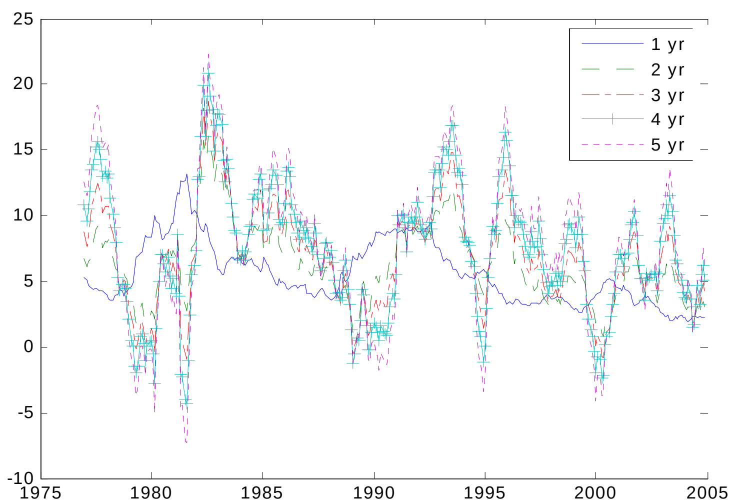
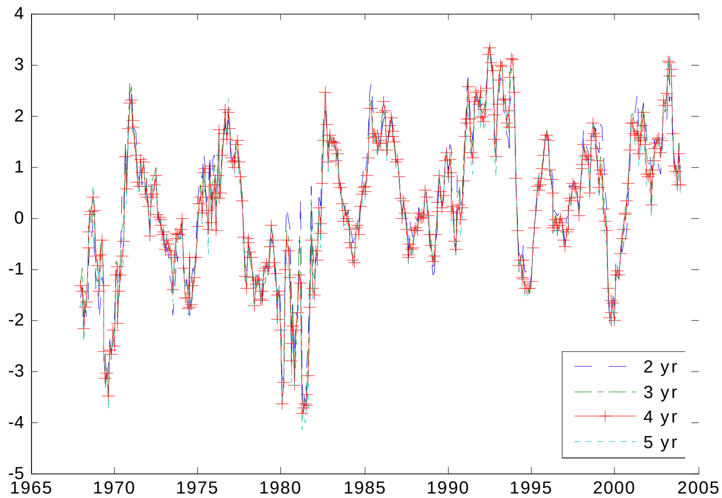


Figure 2. Time Trend in Sharpe Ratios for Bonds with Maturity of 2 through 5 Years

Panel A: U.S. Government Bonds



Panel B: German Government Bonds

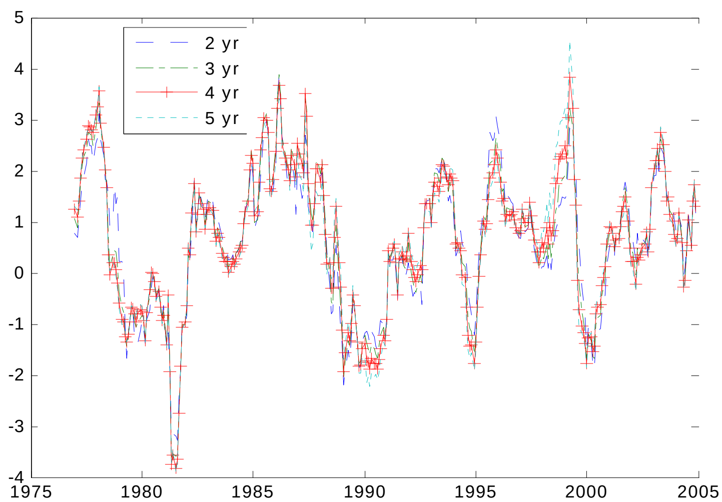
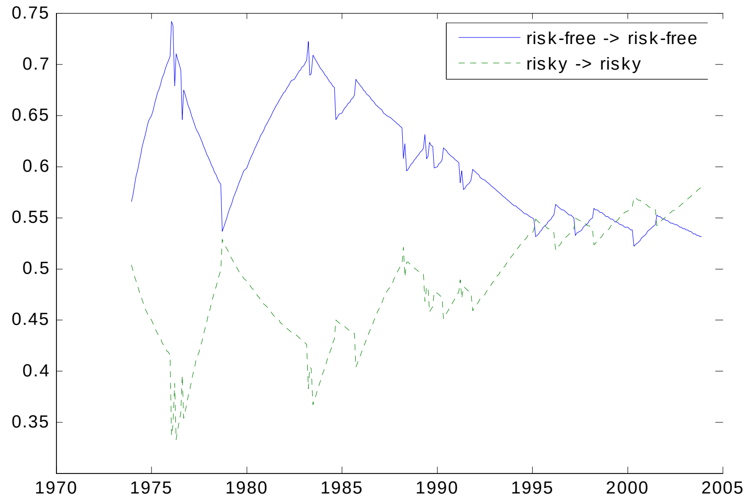


Figure 3. Transition Probability of Markov Process

Panel A: U.S. Government Bonds (1974 to 2003)



Panel B: German Government Bonds (1983 to 2004)

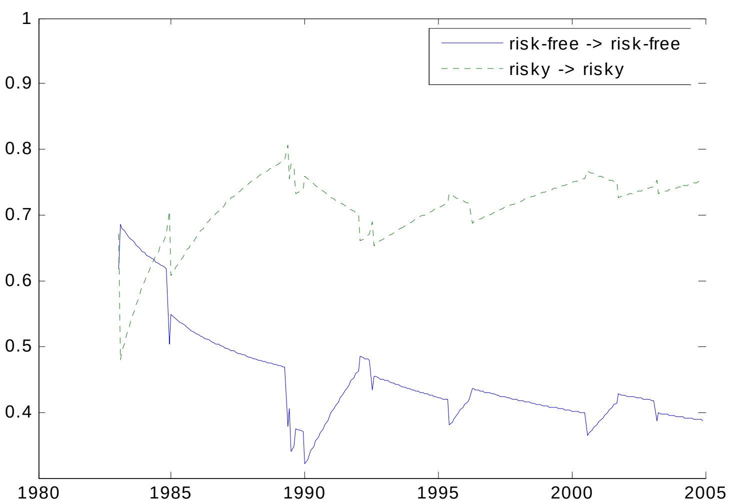
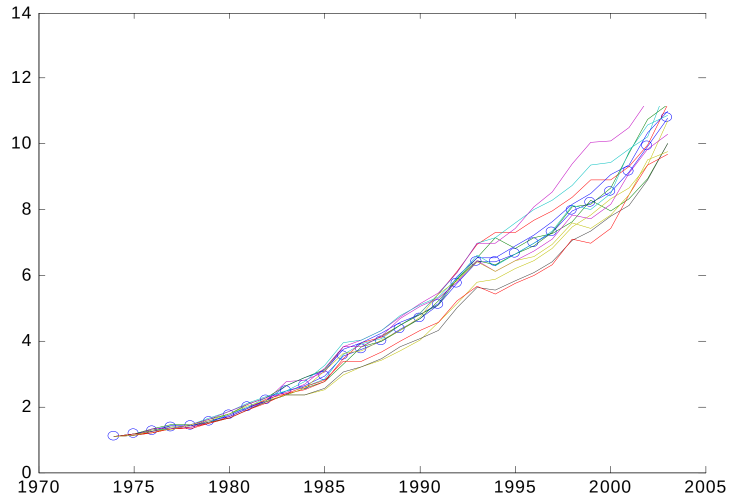


Figure 4. Cumulative Returns under Markov Chain Approach

Panel A: U.S. Government Bonds (1974 to 2003)



Panel B: German Government Bonds (1983 to 2004)

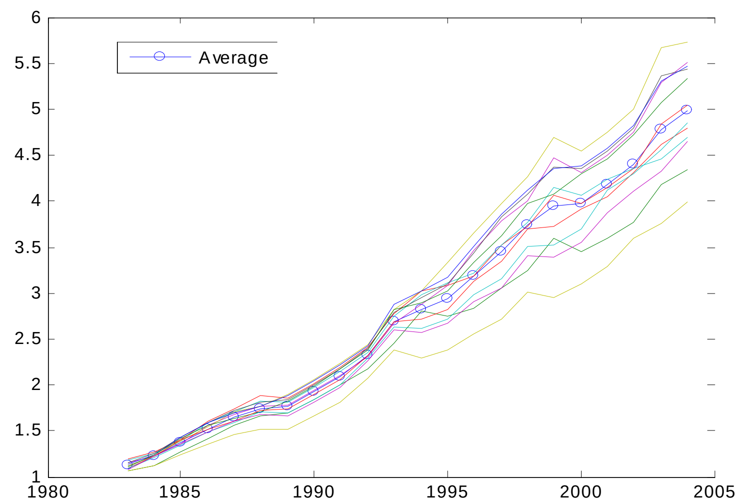
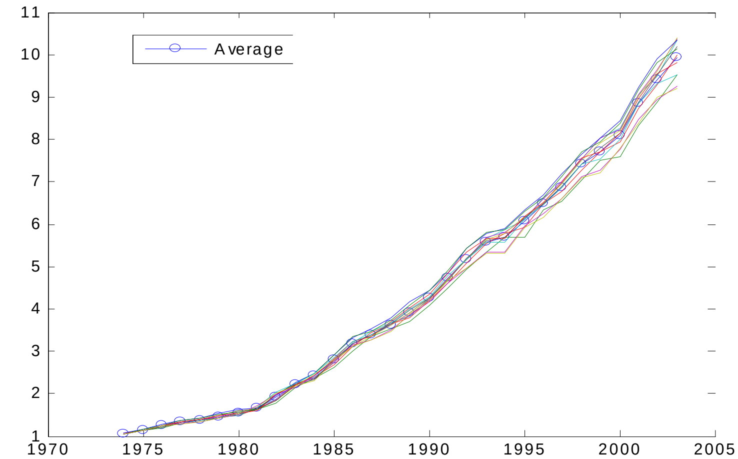


Figure 5.1. Cumulative Returns under Equally Weighted Allocation Method: U.S. Government Bond

Panel A: Cumulative Returns under Equally Weighted Allocation Method



Panel B: Difference in Cumulative Returns Between Markov Chain Approach and Equally Weighted Allocation Method

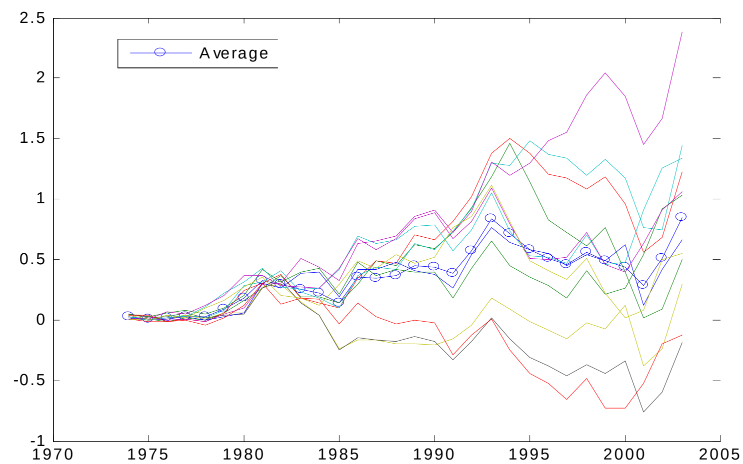
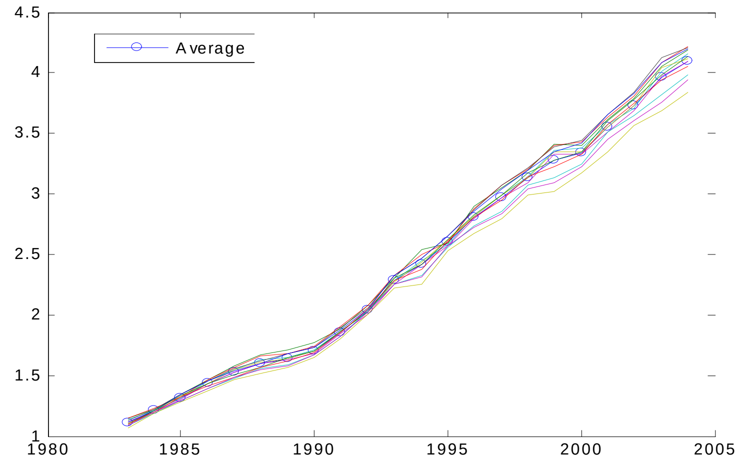


Figure 5.2. Cumulative Returns under Equally Weighted Allocation Method: German Government Bond

Panel A: Cumulative Returns under Equally Weighted Allocation Method



Panel B: Difference in Cumulative Returns Between Markov Chain Approach and Equally Weighted Allocation Method

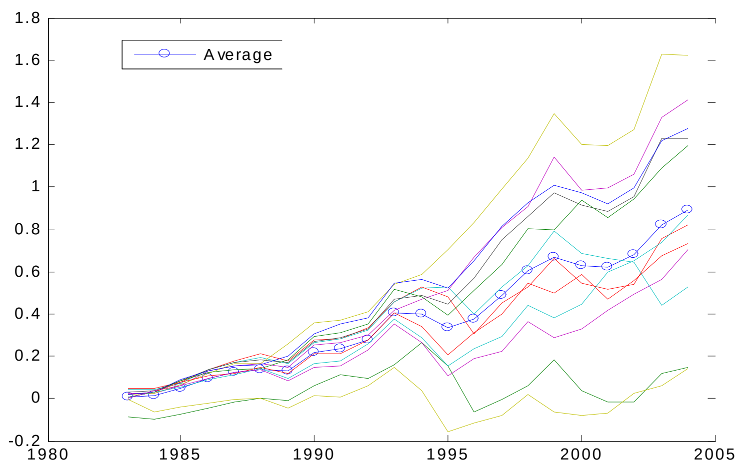
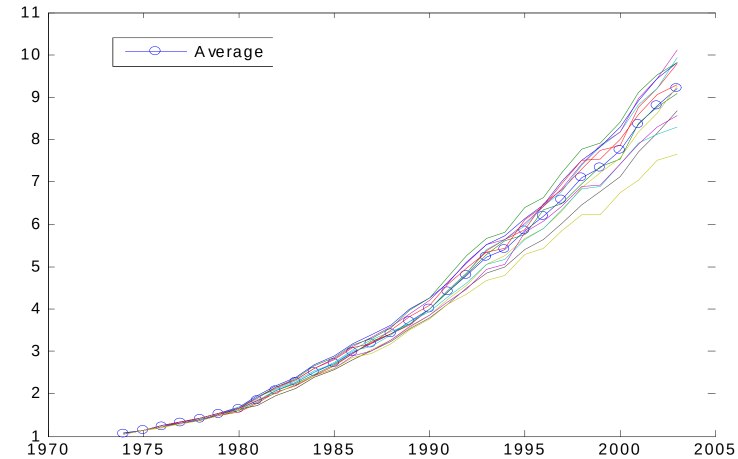


Figure 6.1. Cumulative Returns under Historical Mean Approach: U.S. Government Bond

Panel A: Cumulative Returns under Historical Mean Approach



Panel B: Difference in Cumulative Returns Between Markov Chain Approach and Historical Mean Approach

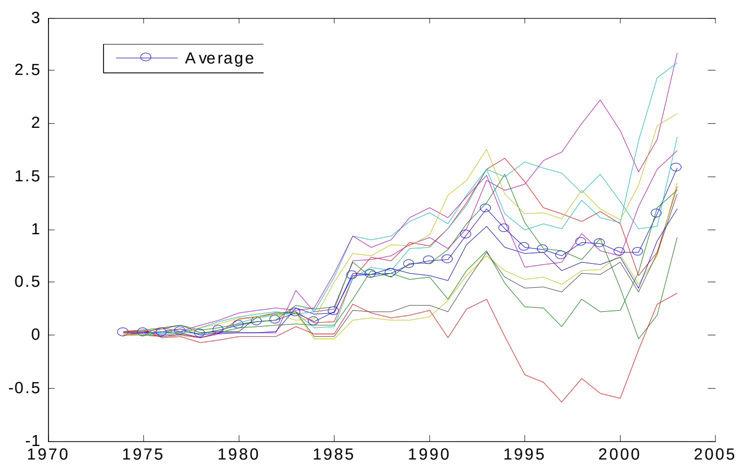
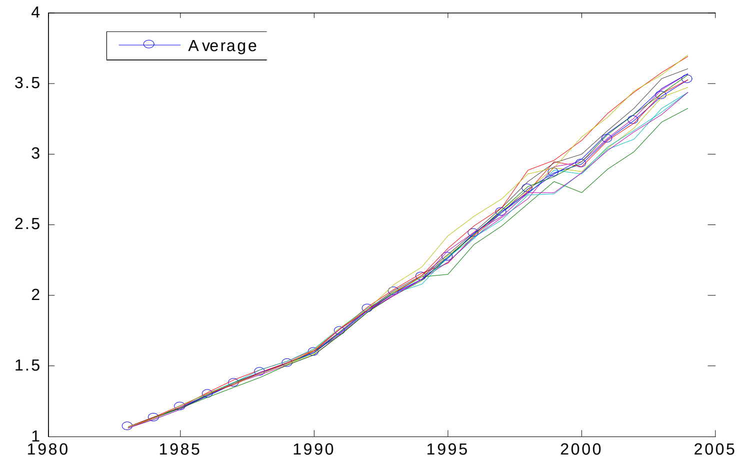


Figure 6.2. Cumulative Returns under Historical Mean Approach: German Government Bond

Panel A: Cumulative Returns under Historical Mean Approach



Panel B: Difference in Cumulative Returns Between Markov Chain Approach and Historical Mean Approach

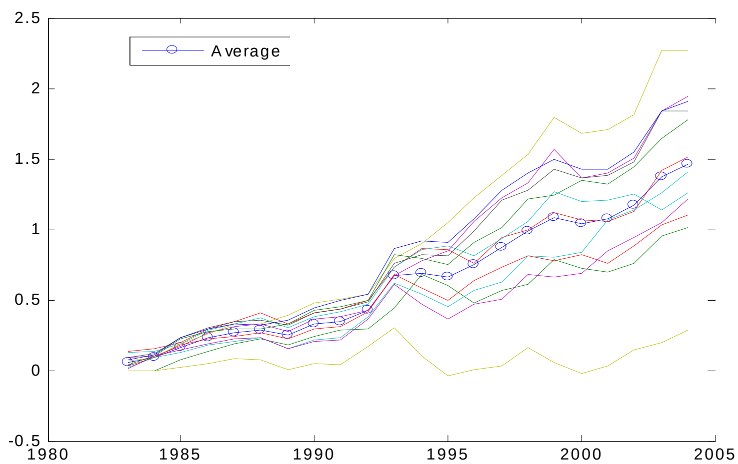
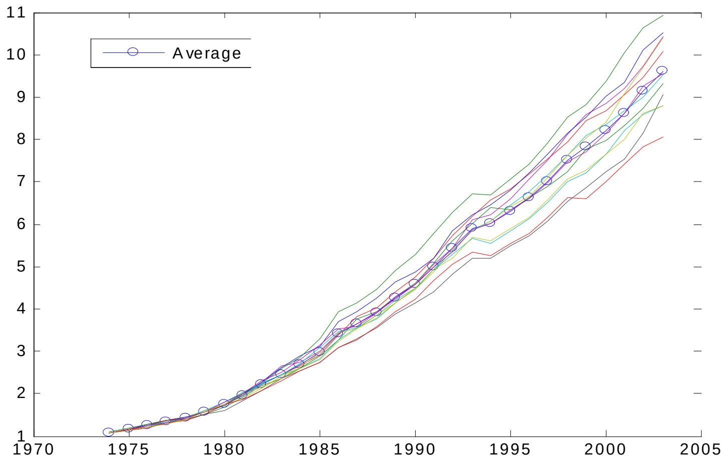


Figure 7.1. Cumulative Returns under Random Walk Approach: U.S. Government Bond

Panel A: Cumulative Returns under Random Walk Approach



Panel B: Difference in Cumulative Returns Between Markov Chain Approach and Random Walk Approach

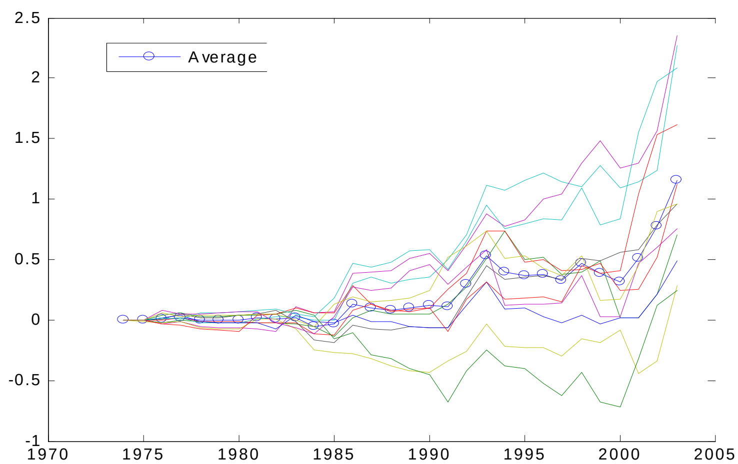
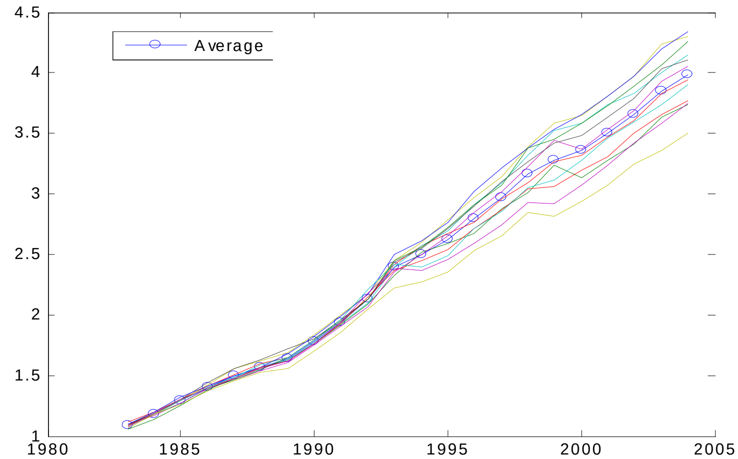


Figure 7.2. Cumulative Returns under Random Walk Approach: German Government Bond

Panel A: Cumulative Returns under Random Walk Approach



Panel B: Difference in Cumulative Returns Between Markov Chain Approach and Random Walk Approach

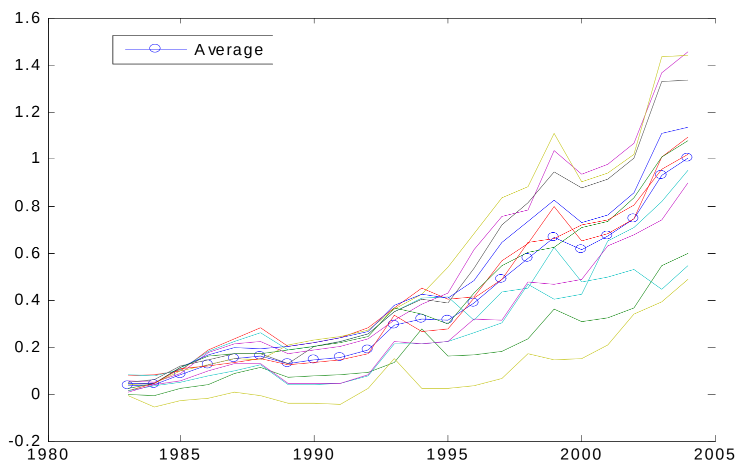
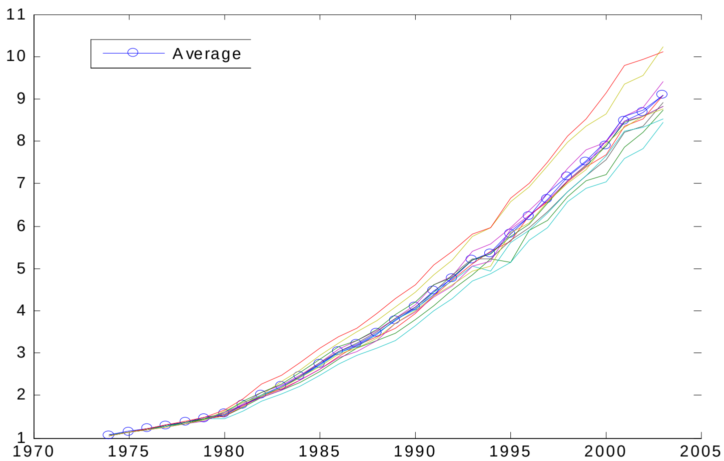


Figure 8.1. Cumulative Returns under AR(1) Approach: U.S. Government Bond

Panel A: Cumulative Returns under AR(1) Approach



Panel B: Difference in Cumulative Returns Between Markov Chain Approach and AR(1) Approach

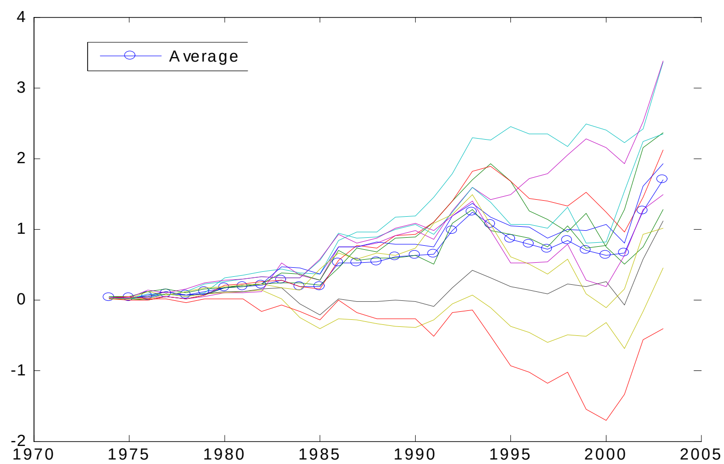
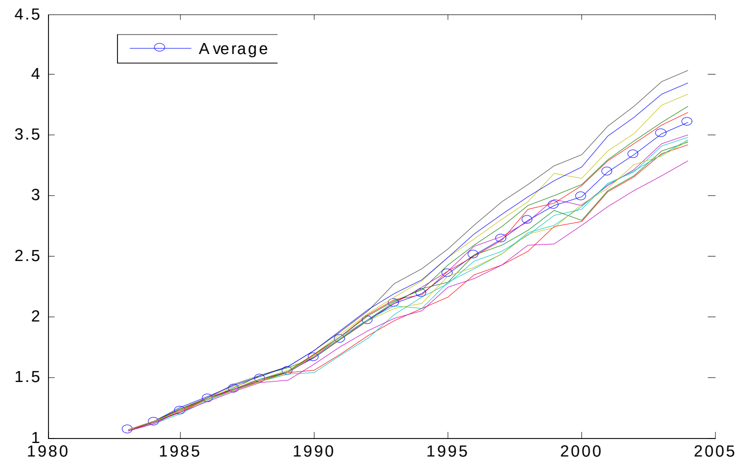


Figure 8.2. Cumulative Returns under AR(1) Approach: German Government Bond

Panel A: Cumulative Returns under AR(1) Approach



Panel B: Difference in Cumulative Returns Between Markov Chain Approach and AR(1) Approach

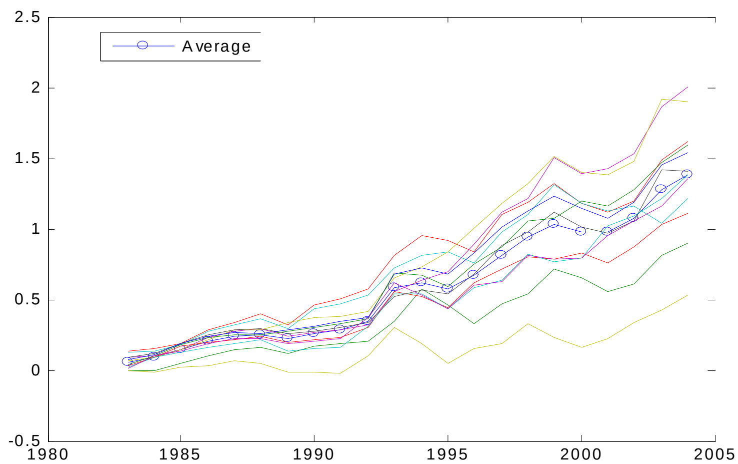
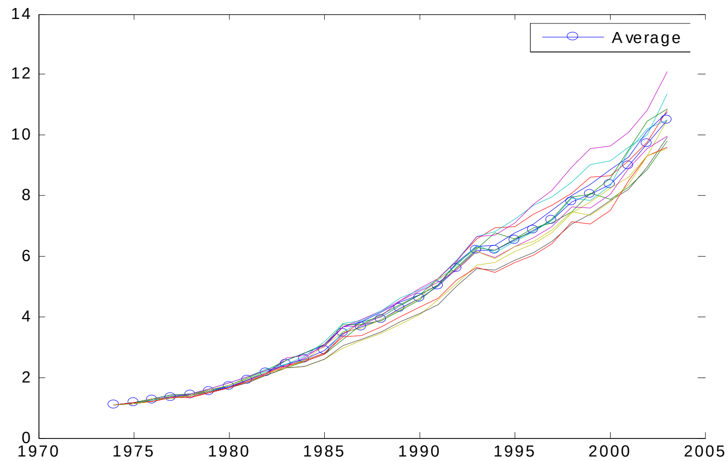
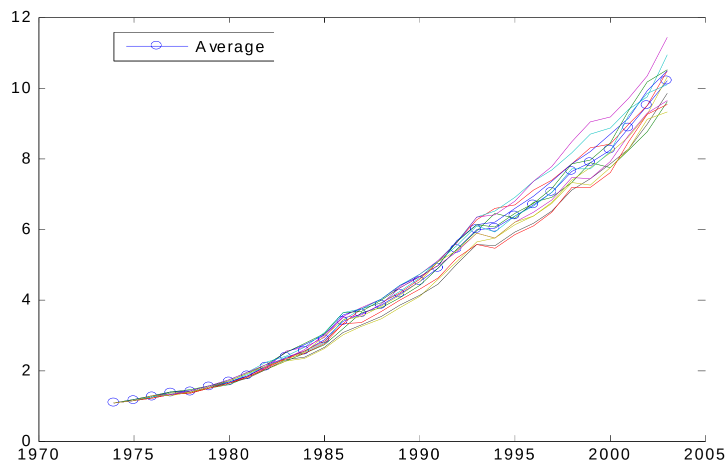


Figure 9.1. Cumulative Returns under MC Approach: U.S. Government Bond

Panel A: Portfolio Restriction $[u, v] = [.1, .9]$



Panel B: Portfolio Restriction $[u, v] = [.2, .8]$



Panel C: Portfolio Restriction $[u, v] = [.3, .7]$

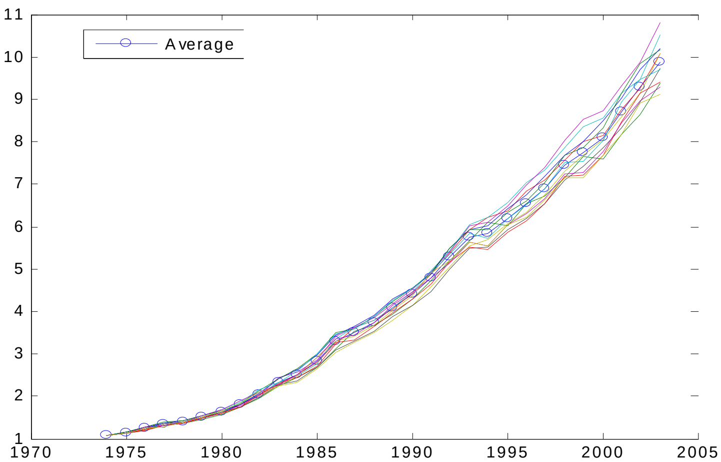
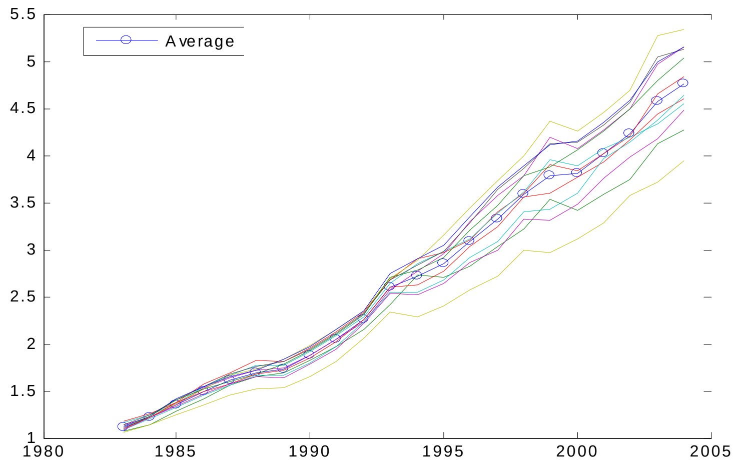
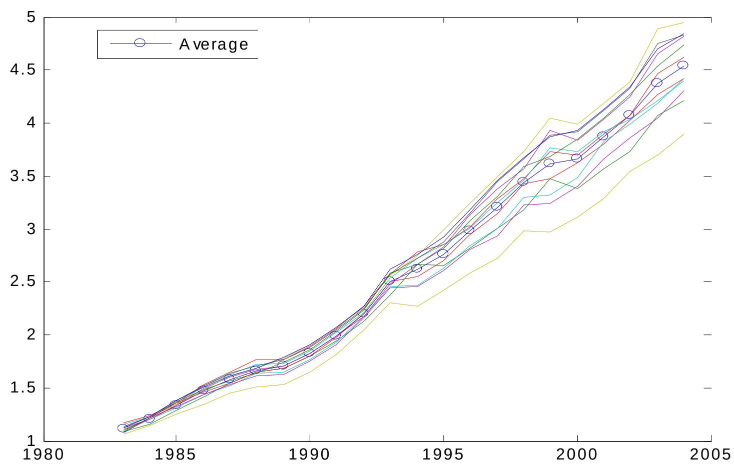


Figure 9.2. Cumulative Returns under MC Approach: German Government Bond

Panel A: Portfolio Restriction $[u, v] = [.1, .9]$



Panel B: Portfolio Restriction $[u, v] = [.2, .8]$



Panel C: Portfolio Restriction $[u, v] = [.3, .7]$