Distribution-Dependent Value of Money: A Coalition-Proof Approach to Monetary Equilibrium

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Abstract

We present a simple, finite-state search model to understand how the cross-sectional distribution of money affects its value. We first document a *network effect*: the value of a given unit of money is higher when its distribution is even, rather than skewed. We also find some distributions to be destabilizing: there is strong incentive to form *coalitions* to "repudiate the incumbent and re-issue new currency" when the distribution is skewed. In this regard, we suggest that conventional "Nash" monetary equilibria be refined to be "coalition-proof" in the spirit of Bernheim et al. (1987). Our approach highlights the merits of investigating non-stationary distributions *per se*, as opposed to (the typically favored) steady states. This approach is designed to be especially pertinent in the context of private issuance of money, in particular, *cryptocurrencies*.

Key Words: cryptocurrency, distribution of money, coalition-proofness, search-theoretic monetary model.

JEL Classification: G10, E00, D83.

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1 Introduction

From commodity to paper money, and more recently to discussions on digital money, the physical form of money has been shaped by the available technology. In particular, recent developments in information technology has ushered in the potential for money to take the form of electronic ledgers, also commonly known as "blockchains". One defining property of this technology is that, in principle, participation in the process of private issuance (for instance, by "mining") is open to all. This lowered cost of entry has consequently rekindled[1] interest in the possibility of privately-issued money in the form of *cryptocurrencies*.

The following two key features of cryptocurrencies are well-known. Firstly, the cross-sectional distribution of a cryptocurrency is typically heavily skewed: a small group of people account for the vast majority of the currency in existence.[2] This reality is in stark contrast to the standard setup in the monetary literature where the cross-sectional distribution of money is typically either assumed normal, or is rendered degenerate. Secondly, the cryptocurrency market has witnessed a surge in the number of entrants since the introduction of Bitcoin (2009), presumably at least in part owing to the lowered entry barrier. The plenitude of entrants, together with the very nature of the technology[3], allows for an environment that is conducive to strategic interactions, yet this possibility has largely been overlooked in the monetary literature as well.

We construct a monetary model that responds to these recent evolutions in technology. Namely, we accentuate the role of the cross-sectional distribution of money by allowing it –and the consequent values of money– to be non-stationary and non-degenerate in our model. Based on this, we furthermore allow the model to accommodate strategic interactions among those that make issuance decisions. These two features of the models are, in fact, intricately linked at the most intuitive level. As an example, consider an extreme distribution where the cryptocurrency in circulation is being held in disproportionately large quantities by only a few individuals. It is then easy to intuit the incentive to form a coalition –comprising of individuals who own less money than the dispro-

[1] Private issuance has been of interest in the academic discourse even before the advent of cryptocurrencies, a recent example being that of Martin and Schreft (2006) among others.

[2] For example, 94.9% of all Bitcoins in existence are held by the top 2.5% addresses as of November 9, 2020. These top holders are typically those who were involved in the creation of coins themselves, and/or have superior mining capacity.

[3] Under the new technology, it is possible to introduce new currency without physically delivering it to individuals, and would be equally possible to retreat an existing currency almost instantly (with a "few line of codes") under the consent of the coalition as long as it is accepted to be beneficial to its members.

portionate few– to repudiate the incumbent currency and issue their own. While such a coalition under a privately issued currency scheme may have been a mere theoretical possibility thus far, the recently developed technology makes it realistically viable, by significantly reducing the associated transaction costs and facilitating smooth communications among prospective colluders. And conversely, it is equally plausible that the benefit from such type of joint deviation would nearly vanish if the distribution is approximately uniform. Hence, taken together, it is evident that the incentive for strategic action hinges critically on the cross-sectional distribution of money. Viewed this way, we can understand the plethora of cryptocurrencies we observe in reality as repeated attempts to jointly deviate from an unequal money distribution; that is, the equilibrium we observe is not "coalition-proof". These considerations motivate us to make a distinction between monetary equilibria that are "coalition-proof" in the spirit of Bernheim, Peleg, and Whinston (1987) –which we call the *voting-proof Nash equilibrium* (VPNE)– and those that are not.

Our model closely mirrors this intuition. It is built on the foundation of standard monetary models in the tradition of Kiyotaki and Wright (1989), but departs from its conventional treatment in two steps. In the first step – which we call the "Nash" model– we allow the distribution of money to be non-stationary *and* publicly known[4], while suppressing the ability to form coalitions. This suppression – and hence the undivided focus on variations in the distribution itself– allows us to clearly understand how the cross-sectional distribution of money affects its value. The "Nash" model reveals a *network effect* of money. Namely, a given unit of money is worth more when the cross-sectional distribution is more uniform, rather than when it is concentrated. For example, consider an economy populated by ten agents with ten units of money in total ($N = 10$). The "Nash" model quantitatively shows that the value of holding the same amount of money (for example, $m = 1$) depends on whether the money stock is distributed *un*evenly (for example, one individual monopolizing 9 units of money) or evenly across the economy (for example, 1 unit of money each), with a clear preference for the latter. This result is intuitive in light of the fact that the value of money in monetary models fundamentally draws from the reduction of search frictions (i.e., "double coincidence of wants"), and that the ability to reduce frictions depends on how widely it is circulated and accepted in every corner of the economy.

[4]This setup is completely realistic in the context of cryptocurrencies and electronic (digital) ledger technology. The distribution –albeit not the identity– is public information that can be easily searched online.

In the second step –which we call the "joint deviation" model– we release the suppression imposed on the "Nash" model and allow for deviation by coalition. While such joint deviation can, in theory, come in all shapes and sizes, we suggest one particular mechanism where agents deviate by *voting*. More specifically, we allow agents to vote at each period, upon whether they wish to continue with the incumbent currency or if they wish to "repudiate and reissue", and if the predetermined quorum (Q) is reached, the economy retreats the currency in circulation and re-issues a new currency which is distributed *equally* among its constituents. We adopt this particularity for the sake of simplicity and focus, and to provide a contrasting alternative to the very motive for deviation; the disproportionate concentration of money. The "joint deviation" model results validate our conjecture on the instability of conventional monetary equilibria. Namely, the equilibrium voting strategy is, indeed, to "repudiate and reissue" when the concentration of money distribution exceeds a threshold level. As such, we define a refinement (i.e., a proper subset, called "voting-proof" Nash equilibrium) of the conventional "Nash" monetary equilibrium as those that can*not* be voted away in our joint deviation model.

Within the class of search-theoretic models of money pioneered by Kiyotaki and Wright (1989, 1991, 1993) and extensions thereon, ours is closely related to those with *non-stationary* money distributions (under a general upper bound on money holdings), most prominently, Green and Zhou (1998) and Berentsen (2002). Models with non-stationarity are typically concerned with *eventual* consequences, such as whether an initial distribution converges to a steady-state distribution. On the other hand, our focus is directly on the non-stationarity process itself, because we are interested in its impact on the value of money and on the subsequent incentive to jointly deviate, a distinguishing feature of our model. This direct focus, while desirable, has been considered a challenge in the literature due to the onerous task of keeping track of individual choice under the ever-changing distribution, as well as its impact back on the distribution itself[5]. We sidestep this difficulty by encoding the distribution directly into the states themselves[6] while keeping the number of agents finite. This definition of states prove judicious enough to allow us to draw economic implications.

[5]Hence the vast majority of the models in the literature have focused on steady states or evolution towards steady state distributions. However, for the purpose of probing the feasibility of private issuance, it is more sensible to investigate whether the process leading up to the steady state is tenable, which is the primary interest of our paper.

[6]This is in sharp contrast to the dominant practice in the literature where states represent the number of units of money possessed by an agent. In such a setup, stationarity and steady states become a natural requirement, so as to fix state-contingent values. Our approach obviates this requirement by letting the states themselves be contingent on distributions, thereby relieving the need for stationarity to compute the value function. This is also an entirely realistic feature in the modern context, since the distribution of cryptocurrency is public knowledge online.

Our work is also related to the literature on refining the Nash equilibria to those that are immune to joint deviations[7]. For example, Aumann (1959) suggests that a Nash equilibrium be called *strong* if and only if it is robust to every conceivable coalition (*Strong Nash Equilibrium*), a notion criticized by Bernheim et. al. (1987) for being "*too strong . . . (to the point that it) almost never exists*". Instead of eliminating every such equilibrium, Bernheim et. al. propose a concept –*Coalition-Proof Nash Equilibrium*– that eliminates only those that can be jointly deviated away in an internally consistent ("self-enforcing") way, namely, the deviations themselves must be impervious to deviations from within[8]. While we recognize the appeal of this notion, it is rather unwieldy, as the concept is defined recursively. We suggest a voting mechanism as an operational simplification that befits our purpose, while carrying the same core message as that of Coalition-Proof Nash Equilibrium, which is to eliminate unsustainable Nash equilibria in an internally consistent manner[9].

We end this section with some realistic implications on cryptocurrency. The network effect of the "Nash" model suggests that the *value* of money –and presumably the success of money issuance scheme– critically relies on the evenness of its distribution, yet this form of "profit sharing" from the viewpoint of the developers would most likely undermine the incentive to develop and issue money in the first place[10]. In addition, our "joint deviation" model raises the issue of *stability*. The hoarding of coins by its creators or miners, while perhaps myopically in their best interest, is likely to be detrimental to their own long-term sustainability as it provides an incentive for the underprivileged others to coalesce in negation of its very own existence. Taken together, these issues may represent fundamental obstacles in the way of success of private issuance of money, especially when compared with its public counterpart[11]. However, we also view our results as constructive advice for future designers of cryptocurrencies, in the sense that one may find an "optimal level of profit-sharing" that strikes a balance between alleviating the value/stability problems while simultaneously securing

[7]Take the classic example of the 2-agent Prisoner's Dilemma game. While the well-known equilibrium consists of the $\{confess, confess\}$ strategy profile, this Nash equilibrium, famously, is not robust to communication among the prisoners. Hence this equilibrium must be refined if we allow for the possibility of deviation in groups.

[8]Consequently, in their definition, more Nash equilibria survive compared to the Strong Nash equilibrium of Aumann.

[9]A voting mechanism eliminates the onus of having to search for all sub-deviations, by dint of the fact that, in our model, if the vote could not attain the quorum in a proposed joint deviation, neither can of any of its sub-joint deviations.

[10]This is especially so, as the total supply must be reigned in to avoid an over-issuance problem. For example, the total supply of Bitcoins that can ultimately be "mined" is capped from above.

[11]For example, central bank digital currency (CBDC) is, by definition, immune to these issues.

some level of profitability for developers[12], although this analysis is beyond the scope of this paper.

2 The "Nash" Model

In this section we introduce the baseline model with*out* the possibility to form coalitions. Since the model, at this stage, does not allow for deviation by coalition, any conceivable deviation is unilateral and we hence call this the "Nash" model[13]. The model is intentionally simplistic to focus on analysing the headline features of the electronic ledger technology used in money. The consequences of coalescence will be explored in sections that follow.

2.1 The Physical Environment

Time is discrete, running from zero to infinity: $t = 0, 1, 2, \ldots$. The economy consists of N individuals, where each agent can either consume or produce one perishable and indivisible good in each period. Following the 'coconut' setup of Diamond (1981), the goods are indistinguishable but agents cannot consume her own production; namely she *must* trade to consume. Production and consumption is instantaneous. At each period, each member of the economy is assigned to one of $\frac{N}{2}$ trading pairs ($\frac{N-1}{2}$, if N is odd) by nature. Each trading pair consists of one consumer (buyer) and producer (seller), also assigned by nature. The *trading pair assignment rule* is that all assignments are equally likely, and is determined independently for each and every period, a rule that is known to all agents. Note that this setup precludes bartering: it is not possible to agree to "pick coconuts for each other" as there is only one producer within the trading pair. Preferences are identical across agents, hence all equilibria are symmetric. Each agent enjoys utility of u from every unit of consumption and incurs a cost of c from production of every unit of good, with $u > c > 0$. Future utility and cost are discounted by $\beta \in (0, 1)$ per period.

There are N indivisible units of money in circulation. Since agents can neither commit to future actions nor are the full trading histories available in our setup, money naturally emerges as a

[12] We envision the feasibility of such a compromise in light of the fact that the root cause of these issues is ultimately the concentration of money distribution, which is also a source of profitability for developers. Hence, it may be optimal for the developers to voluntarily relinquish the hoarding motives to a certain extent in order to ensure stability and value. Realistically, this could take the form of designing an initial coin offering (ICO) rule that is conducive to a more widespread distribution.

[13] The name draws from the fact that the deviations that are considered from Nash equilibria are typically individual deviations (say, for example, deviations from the Prisoner's Dilemma game), rather than in coalition with other players.

medium of exchange[14]. While the number of units of money in circulation could be set to be an arbitrary finite number, we choose it to equal the number of agents to anchor the value of money to the "nominal GDP" [15] as the effect of increasing this upper bound has been previously explored (see, for example, Taber and Wallace 1999). Other than this bounded supply, there is no upper limit on the units of money owned by an individual; in an extreme case, one agent can own all N units of money, and in the other extreme, all N agent may hold one unit each. In order to simplify the model, we impose a carry constraint on money following Berentsen (2002); agents can only take one unit of money to trade. Together with the assumed indivisibility of money and goods, this obviates any fluctuation in the price of goods.

2.2 The Timeline and Information Structure

Quite naturally, the agent knows her own stock of money at any point in time, denoted m_t. It is also assumed that the cross-sectional money distribution –albeit not the identity of the holders– of the economy is always public information. This assumption is primarily for realism, since this feature is a key defining property of cryptocurrencies as an openly distributed, cross-verifiable ledger. Making this knowledge public also helps us elucidate how the *distribution* of money can affect its value, which is one of the main goals of our model. For a given N, we denote the set of all possible money distributions by $\mathcal{D}_N$, and its time-t-realization, by $\Delta_t \in \mathcal{D}_N$. For example, when $N = 3$, $\mathcal{D}_3 = \{(0,0,3),(0,1,2),(1,1,1)\}$. If $\Delta_t = (0,0,3)$, this would denote the situation where two agents in the economy are penniless and one agent holds all three units of money, etc[16]. Our assumption requires that $\Delta_t \in \mathcal{D}_N$ is public knowledge at t.

Consider a generic time period, $[t, t+1)$. As assumed, the agents walk into t knowing their own money stock (m_t) and the cross-sectional money distribution (Δ_t). We denote the information available at t as $\Omega_t := \{m_t, \Delta_t\}$. Then, at an arbitrary interim period, denoted '$t+\epsilon$', nature draws the trading pairs as dictated by the assignment rules and announces the pair to each individual. Each agent then knows who she is matched with, as well as whether she is to be a consumer

[14] According to Araujo (2004), the essentiality of money can also arise when N is large. However we keep N low for computational tractability.

[15] Models of privately issues money with no such upper bound have been explored, and largely been concluded infeasible because of time inconsistency issues. (See, for example, Ritter 1995 and Taub 1985.) Namely, if issuance is essentially costless and unlimited, money will be issued ceaselessly until its value is inflated away to zero. Some cryptocurrencies seem to have recognized this problem and responded by placing an upper bound, providing a natural validity to our setup.

[16] Note that all agents are identical, except for their possession of money, hence we do not distinguish between say, $(0,0,3)$ and $(3,0,0)$.

(buyer) or producer (seller). We denote this consumer/producer assignment by $\chi \in \{C, P\}$, where 'C' denotes consumer and 'P' denotes producer. Once the trading pair is set, it is assumed that those within the pair can observe her partner's stock of money, n_t, as well. This setup is standard in the literature. Note that while the agents know their *own* trading pair, they do not know the identities, or the specific money stock of *other* trading pairs.[17] In short, at '$t + \epsilon$', each agent knows (1) $\Omega_t := \{m_t, \Delta_t\}$, (2) the trading pair she is assigned to, and in particular, her partner's money stock (n_t), and (3) whether she is to be the producer or consumer ($\chi \in \{C, P\}$). We let $\Omega_{t+\epsilon} := \{\Omega_t\} \cup \{\chi_t, n_t\} = \{m_t, n_t, \chi_t, \Delta_t\}$ denote this augmented information available at '$t + \epsilon$'.

Based on these information, the agent implements her strategy –detailed further in the subsection that follows– while trading with her partner at '$t + \epsilon$', incurring an instant payoff (Φ) of either 0, u, or $-c$. To keep the book-keeping tidy, we discount the payoffs ($\{u, -c, 0\}$) as if they are delivered and consumed/defrayed at 't', as '$t + \epsilon$' is just a nominal time period we introduce for exposition.[18] This concludes all action prior to $t + 1$. The time-t states (Ω_t), together with the trades that were implemented during '$t + \epsilon$', jointly determine the money stock of each and every individual in the next period (m_{t+1}), and consequently, the cross-sectional distribution of money in the next period (Δ_{t+1}) as well. As agents walk into $t + 1$, Δ_{t+1} becomes public knowledge and hence $\Omega_{t+1} := \{m_{t+1}, \Delta_{t+1}\}$ is privately known to each agent. The same process described above is then re-iterated at $t + 1$, and consecutively in all time periods that follow.

2.3 The Agent's Problem and Strategy

The agent seeks to maximize the future expected stream of payoffs

$$V_t(\Omega_t) = \mathbb{E}_\pi \left[\sum_{s=t}^{\infty} \beta^{s-t} \Phi_\pi \left(\pi^\chi(\Omega_{s+\epsilon}) \right) \cdot \mathbb{1}_F(\Omega_{s+\epsilon}) \middle| \Omega_t \right] \tag{1}$$

at any given time $t \geq 0$. $\pi^\chi(\Omega_{s+\epsilon}) := \pi^\chi(\{m_s, n_s, \chi_s, \Delta_s\})$ denotes the strategy that is to be implemented during time node $[s, s + 1)$, $s \geq t$. We restrict our attention to pure strategies only. This restriction can be thought of as a setup where agents agree to trade only if it provides a *strict* increment in expected discounted utility. The strategy (π^χ) is indexed by $\chi \in \{C, P\}$, i.e., whether she is the designated consumer (C) or producer (P) in that period. Since the setup does not allow

[17] It is still possible to infer the *distribution* of money stock of other trading pairs from Δ_t and n_t, which we take into account when we set up the Markov transition matrix in the sections that follow.

[18] We might as well have assumed that payoffs are delivered at '$t + 1$', however this only changes the timing of discount and does not lead to any meaningful difference in results.

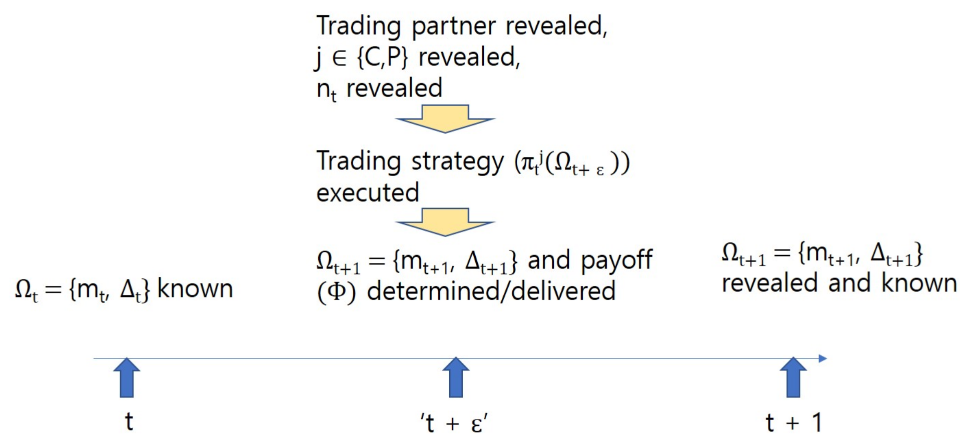

Figure 1: A representative time node [t,t+1)

for bartering, the only feasible trade is either to purchase a good in exchange for a unit of money (if $\chi = C$):

$$\pi^C(\{m_s, n_s, C, \Delta_s\}) = \begin{cases} 1, & \text{if she agrees to purchase a good in exchange of 1 unit of money} \\ 0, & \text{otherwise,} \end{cases}$$

or to produce and sell a unit of good in exchange for a unit of money (if $\chi = P$):

$$\pi^P(\{m_s, n_s, P, \Delta_s\}) = \begin{cases} 1, & \text{if she agrees to produce \& sell a good in exchange of 1 unit of money} \\ 0, & \text{otherwise.} \end{cases}$$

Note that this binary setup is in tandem with our assumed focus on pure strategies only.

Given this strategy profile, the expectation is evaluated under the belief that this strategy will be implemented, which indeed is the case in equilibrium. Also, $\left\{\Phi_\pi\big(\pi^\chi(\Omega_{s+\epsilon})\big)\right\}_{s=t}^{\infty}$ denotes the stream of payoffs under this strategy profile, where $\Phi_\pi\big(\pi^\chi(\Omega_{s+\epsilon})\big) \in \{u, -c, 0\}$, $\forall s \geq t$. Clearly, the realization of payoffs (Φ) in each period depends not only on the agent's *own* strategy profile ($\pi^\chi(\Omega_{s+\epsilon})$) but on those of others because a trade can go through only if '$\pi^C(\Omega_{s+\epsilon}) \times \pi^P(\Omega_{s+\epsilon}) = 1$' holds within a trading pair. And since we know that the equilibrium will be symmetric, we generically denote this symmetric strategy as 'π'[19] and index the payoff by this symmetric strategy: Φ_π. Similarly, the expectation operator is indexed by π.

[19] We will provide a more precise expression of this function in the subsection that follows after introducing a few notations.

$\mathbb{1}_F(\Omega_{s+\epsilon})$ is an indicator that takes note of the fact that trades are not feasible if the consumer has no money to spend. This can happen if the agent is assigned to be the consumer ($\{\chi = C\}$) and has no stock of money ($m_s = 0$), *or* if the agent is assigned to be the producer ($\{\chi = P\}$) and her trading partner has no stock of money ($n_s = 0$). This feasibility condition is formalized in the following notation:

$$\mathbb{1}_F(\Omega_{s+\epsilon}) = \begin{cases} 0, & \text{if } \Big(\{m_s = 0\} \cap \{\chi = C\}\Big) \cup \Big(\{n_s = 0\} \cap \{\chi = P\}\Big) \\ 1, & \text{otherwise.} \end{cases}$$

Finally, $\beta \in (0, 1)$ is the time discount factor that discounts all future payoffs.

We end with a remark on the structure of the timeline and the state variable. We have consciously chosen the time-t information (Ω_t) to be, in fact, the *minimal* Markov state variables necessary for the goal of our model, which is to elucidate how $\Omega_t := \{m_t, \Delta_t\}$ jointly affect the value of money and generate incentives to form coalitions. This means, however, that time-t information (Ω_t) is not sufficient to execute trading strategies $\pi^\chi(\Omega_{t+\epsilon})$ since $\Omega_t \subsetneq \Omega_{t+\epsilon}$, and hence, the strategies must be executed in the interim period '$t + \epsilon$'. Those who find this slightly unusual can think of this setup as committing[20] to a complete contingency plan at time-t, and mechanically executing the plan as new information ($\{\chi_t, n_t\}$) is revealed to them at '$t + \epsilon$.' Hence, $V_t(\Omega_t)$ can be interpreted as the equilibrium value of the entire collection of contingency plans that is to be executed in the future ($s > t$), given the current state Ω_t.

2.4 The Markov Chain and its Transition Probability Matrix

The variables in $\Omega_t := \{m_t, \Delta_t\}$ are jointly Markov[21], which we henceforth define as states. We first belabor the definition and notation of states, as its construction is a subtle but important novelty of our model. For any given N, there is a combinatorically finite number of cross-sectional distribution states, the entire set of which we denote as $\mathcal{D}_N$. Somewhat against the common convention of expressing each element as a "probability mass function" – i.e., as the frequency of agents holding a given number of money– we choose instead to directly express the number of

[20]This commitment is credible in equilibrium, given that the incentive compatibility conditions hold, which we specify in a subsection that follows.

[21]Since nature's trading pair assignments are assumed independent over time, and the goods/agents are indistinguishable, the past history of distributions or trades are irrelevant in determining the agents' strategies, and hence does not affect the transition probabilities.

money held by agents, unique up to ranking as identity is not important. For example,

$$\Delta = \underbrace{(0,0,0,0,1,1,2,4)}_{\text{units of money held by each agent (ranked)}} \in \mathcal{D}_8$$

denotes the situation where $N(=8)$ units of money are shared by only 4 agents, in particular, one agent hoards 4 units of money. This notation is for the sake of expositional transparency, in line with our focus on tracking the evolution of cross-sectional distribution *per se*[22].

We then enumerate the states of the Markov chain for the given N, by taking the product of all possible distribution states ($\Delta \in \mathcal{D}_N$) with all the individual money states (m) therein, in lexicographic order. We denote the state space of the N-agent Markov chain by Ω_N, and provide an explicit example of Ω_N for $N = 3$ in Table 1. In this example, $\mathcal{D}_N = \{(0,0,3),(0,1,2),(1,1,1)\}$, with a total of three possible distributions ($n(\mathcal{D}_N) = 3$). Within $(0,0,3)$ there are two distinct individual money states that can arise; $m = 0$ and $m = 3$. Notationally, we encircle the individual money state and represent them as $(⓪,0,3)$ and $(0,0,③)$, respectively. Likewise, $(0,1,2)$ allows for three distinct individual money states, $(1,1,1)$ allows for one distinct individual money state, hence there are six (2+3+1 = 6) possible states for the Markov chain ($n(\Omega_N) = 6$). Similarly, we can do the enumeration for any given N. In general, it can be shown that number of states ($n(\Omega_N)$, $N \geq 3$) is given by $n(\Omega_N) = \sum_{k=0}^{N} p(k) - 1$, where $p(k)$ denotes the number of *integer partitions* of k. (See Lemma 5, Appendix.) Since $p(k)$ is always finite, the evolution of states in our model is governed by *finite* Markov chains.

As in any Markov Chain, the states evolve as prescribed by the transition probabilities, which in turn depend on the strategy profile (π). For a given N and π, the transition probability is represented by an $n(\Omega_N) \times n(\Omega_N)$ matrix $P^\pi = [P^\pi_{i,j}]$, where $P^\pi_{i,j}$ denotes the probability of transitioning to state ω_j, contingent on being in the current state ω_i. Intuitively, if the current state is ω_i, and strategy is fixed at π, the only probabilistic element remaining is nature's matching assignment in state ω_i, which we denote as $\tau \in \mathcal{T}_N(\omega_i)$[23]. Hence, $P^\pi_{i,j}$ is simply the sum of measures given to the

[22] Under the "*probability mass function* notation" which is more prevalent in the literature, this would be represented as: $\underbrace{(4,2,1,0,1,0,0,0,0)}_{\text{number of agents with 0,1,...,8 units of money}}$. This is cognitively more cumbersome to translate into individual holdings of money, which plays a more focused role in our model. Of course, the difference is only nominal, as both representations contain the same information content.

[23] $\mathcal{T}_N(\omega_i)$ denotes the set of all possible trading pairs and producer/consumer assignments that can be given by nature when the economy is in state ω_i.

State (ω_i)	Money stock (m)	Distribution (Δ)	Notation
ω_1	0	(0,0,3)	(0,0,3)
ω_2	3	(0,0,3)	(0,0,3)
ω_3	0	(0,1,2)	(0,1,2)
ω_4	1	(0,1,2)	(0,1,2)
ω_5	2	(0,1,2)	(0,1,2)
ω_6	1	(1,1,1)	(1,1,1)

Table 1: Numbering of Markov chain states (Ω) when $N = 3$

collection of matching results $\{\tau\}$ that lead to –under the given strategy profile π– state ω_j. The following expression for $P^{\pi}_{i,j}$ formalizes this notion:

$$P^{\pi}_{i,j} = \sum_{\tau \in \mathcal{T}_N(\omega_i)} \mu(\tau) \cdot \mathbb{1}_{\{\mathbb{T}^{\pi}_{\tau}(\omega_i)=\omega_j\}},$$

where $\mu(\tau)$ denotes the probability measure[24] given to $\tau \in \mathcal{T}_N(\omega_i)$ and $\mathbb{T}^{\pi}_{\tau} : \Omega_N \longrightarrow \Omega_N$ is the "transition mapping" that maps the current state $\omega \in \Omega_N$ to a future state $\omega' \in \Omega_N$, contingent on (i) the matching assignment $\tau \in \mathcal{T}_N(\omega)$, (ii) the given the strategy profile π, and (iii) the feasibility condition (i.e., $\mathbb{1}_F = 1$) described in the previous subsection[25]. Finally, given these notations, we can write down π precisely as '$\pi : \Omega_N \times \{\mathcal{T}_N(\omega_i)\}_{\omega_i \in \Omega_N} \longrightarrow \{1, 0\}$'.

2.5 The "Nash" Equilibrium and Bellman Equations

A (symmetric Nash) equilibrium is a sequence of $\{\Omega_t\}_0^{\infty}$[26], its probability transition matrix P^{π}, and (symmetric) strategy profile $\pi : \Omega_N \times \{\mathcal{T}_N(i)\}_{i \in \Omega_N} \longrightarrow \{1, 0\}$, such that the following hold:

[24]The explicit expression for this measure depends on the trading pair assignment rule and N. For example, in our setup where all assignments are equally likely, $\mu(\tau) = \frac{\frac{N}{2}!}{2\cdot\binom{N}{2}\binom{N-2}{2}\cdots\binom{2}{2}}, \forall\tau$, when N is even and similarly $\mu(\tau) = \frac{\frac{N-1}{2}!}{2\cdot\binom{N-1}{2}\binom{N-3}{2}\cdots\binom{3}{2}}, \forall\tau$ when N is odd.

[25]By construction, these three ingredients are sufficient to characterize ω' uniquely given any current state ω. And since Ω_N contains the entire possible states, $\mathbb{T}^{\pi}_{\tau} : \Omega_N \longrightarrow \Omega_N$ is a well-defined surjection, hence $P^{\pi}_{i,j}$ is well-defined for any $\omega_i, \omega_j \in \Omega_N$.

[26]The additional information contained in $\{\Omega_{t+\epsilon}\}_0^{\infty}$ is redundant here. While the realization of trading pairs *is* a factor that determines the evolution of the states Ω_t, its knowledge is not a requirement in our model insofar as P^{π} correctly describes the evolution of states in equilibrium, which can be done by incorporating the *ex ante* probabilities of trading pair assignments (i.e., ex ante forecasts of the extra information that is to be revealed in $\{\Omega_{t+\epsilon}\}_0^{\infty}$) when we compute P^{π}. In short, it is not necessary to keep track of *how* the Ω_t state evolved from t to $t+1$, as long as the evolution of Ω_t obeys the rule dictated by P^{π}, as we are primarily interested in the states Ω_t and its impact on the value of money.

(i) [*Optimality*] Each agent chooses her strategies to maximize equation (1) at each t,[27] subject to the equilibrium strategies of other agents, the given probability transition matrix P^π and nature's trading pair assignment rules. (ii) [*Feasiblity*] The chosen strategy is executed if and only if it is feasible. (iii) [*Rational Expectations*] The probability transition matrix P^π and the evolution of $\{\Omega_t\}_t^\infty$ is consistent with the equilibrium strategy and nature's trading pair assignment rules.

Our setup is conducive to a Bellman representation, hence (1) can be recast into the following dynamic programming problem:

$$V(\omega) = \max_{\pi} \mathbb{E}_\pi[\Phi_\pi + \beta \cdot V(\omega')|\omega], \;\; \omega \in \Omega_N. \tag{2}$$

Let $\omega_i, \omega_j \in \Omega_N$ which will denote current and next period states, respectively. Then $\{\mathbb{T}^\pi_\tau(\omega_i)\}_{\tau \in \mathcal{T}_N(\omega_i)}$ denotes the collection of all possible states that can emanate from the current state i, given π. Let $m(\omega_i)$ and $m(\omega_j)$ denote the money stock of the agent in each state. Then (2) can be written down more explicitly as:

$$V(\omega_i) = \sum_{\omega_j \in \left\{\mathbb{T}^\pi_\tau(\omega_i)\right\}_{\tau \in \mathcal{T}_N(\omega_i)}} P^\pi_{i,j} \cdot \Big(\underbrace{u \cdot \mathbb{1}_{\{m(\omega_i)-m(\omega_j)=1\}} - c \cdot \mathbb{1}_{\{m(\omega_i)-m(\omega_j)=-1\}}}_{\Phi_\pi} + \beta \cdot \underbrace{V(\omega_j)}_{V(\omega')} \Big), \;\; \omega_i \in \Omega_N, \tag{3}$$

where the strategies, in equilibrium, must satisfy the following "incentive compatibility" (for example, Matsuyama, Kiyotaki and Matsui, 1993) conditions:

$$\begin{aligned} &(\forall \tau \in \mathcal{T}_N(\omega_i) \;\text{ and }\; \forall \omega_i \in \Omega_N), \quad \hat{\pi}(\omega_i, \tau) = 1 \\ &\qquad \text{if and only if} \quad \big(\Phi_{\hat{\pi}} + \beta \mathbb{E}_{\hat{\pi}}[V(\omega_j)|\omega_i, \tau]\big)\Big|_{\hat{\pi}(\omega_i,\tau)=1} > \big(\Phi_{\hat{\pi}} + \beta \mathbb{E}_{\hat{\pi}}[V(\omega_j)|\omega_i, \tau]\big)\Big|_{\hat{\pi}(\omega_i,\tau)=0}. \end{aligned} \tag{4}$$

To clarify the notations used in (4), $\hat{\pi}(\omega_i, \tau)$ denotes a potential (unilateral) deviation strategy[28] pertaining to an individual who is in state $\omega_i \in \Omega_N$ and is given the matching assignment $\tau \in \mathcal{T}_N(\omega_i)$, under the presumption that all others will adhere to π. Note that (4) represents an entire array of conditions as it must hold for each and every combination of $\{\omega_i, \tau\}$, and similarly, (3)

[27]Here, we adhere to our earlier interpretation that the contingent strategies –contingent on the revelation of nature's trading pair assignments– are written down at each t, and its actual implementation is mechanically executed at '$t + \epsilon$', when $\Omega_{t+\epsilon}$ becomes available.

[28]Unlike $\pi : \Omega_N \times \{\mathcal{T}_N(\omega_i)\}_{\omega_i \in \Omega_N} \longrightarrow \{1, 0\}$, $\hat{\pi}(\omega_i, \tau)$ concerns the possibility to deviate within one specific point in domain of the strategy function, $\{\omega_i, \tau\}$. Also, recall that any strategy, and in particular, $\hat{\pi}$ is "$\Omega_{t+\epsilon}$-measurable" in our setup, hence it is clear that $\hat{\pi}$ is well-defined given $\{\Omega\} \cup \{\tau\} \supset \Omega_{t+\epsilon}$. Since $\hat{\pi}$ is $\Omega_{t+\epsilon}$-measurable, $\Phi_{\hat{\pi}}$ can be taken out of the expectation as in (4). Naturally, $\hat{\pi}(\omega_i, \tau)$ may (or may not) deviate from the equilibrium symmetric strategy (π).

represents a system of equations, which can be solved for $\{V(\omega_i)\}_{\omega_i \in \Omega_N}$. Ensuring that a set of $\{V(\omega_i)\}_{\omega_i \in \Omega_N}$ and π jointly satisfy (3) and (4) prevents one-shot deviations, which is in fact sufficient to prevent any deviation altogether (see Howard, 1960 and Abreu, 1988), whence it can be claimed that $\{V(\omega_i)\}_{\omega_i \in \Omega_N}$ and π constitute an equilibrium. Because the deviations we consider are unilateral, this equilibrium is Nash, and we henceforth call this (i.e., the equilibrium with*out* the possibility to coalesce) the "Nash equilibrium" (NE).

2.6 Existence and Multitude of "Nash" Equilibria

We start with a Lemma that ensures the existence of (at least one) Nash equilibrium. Assume "reasonable" parameters, namely those that –in line with the prevalent tradition in the literature[29]– sustain maximal trade ($\pi^{\chi} \equiv 1$). Intuitively, this means the the cost of production c is sufficiently low so as to foster sales of good ($\pi^P \equiv 1$), and that the β is also low enough so that agents will not forgo an opportunity of instant consumption ($\pi^C \equiv 1$).[30]

Lemma 1. *Assume a "reasonable" parameter configuration. Then there exists at least one non-trivial (i.e., not autarky) Nash equilibrium.*

Proof. See Appendix. □

It is important to note that Lemma 1 does not preclude the possibility of multiple equilibria. In fact, under "reasonable" parameters, multiple equilibria is typically the norm. As an example, consider $\omega \in \Omega_N$ where $\pi^{\chi}(\omega) = 1$, $\chi = \{C, P\}$. It is easy to intuit that an interlocking switch-over to $\pi^{\chi}(\omega) = 0$, $\chi = \{C, P\}$ may also be an equilibrium, since there is no reason to deviate unilaterally if the trading partner's strategy is $\pi^{\chi}(\omega) = 0$. This conjecture is indeed born out in our computation results[31], the detail of which is the focus of the next section.

[29] The tradition to focus on monetary equilibria that engender maximal trade is, presumably, to focus on equilibrium where money is maximally beneficial in reducing search friction. However, one novelty of our approach is in showing that this focus may be misguided when the cross-sectional distribution is taken into account. We later show in Section 4 that it may indeed be beneficial to voluntarily turn down trades in order to avoid skewing the money distribution excessively.

[30] The notion of "reasonable parameter values" follows Berentsen (2000). To show that such a configuration (that induces $\pi^{\chi} \equiv 1$) indeed must exist, consider the incentive compatibility conditions of the seller and the buyer with vanishingly low c and β. The result follows by applying IVT. Operationally, for the sake of computation, c and β need not be "vanishingly low", as in evidenced in our Example in 3.2.1 with $\beta = 0.96$ and $c = 0.3$.

[31] Hence, the full set of Nash equilibria is thicker than those typically considered as "monetary equilibria" which implicitly assumes that trades occur whenever it is feasible.

3 Computation and Results: the Nash Equilibrium

3.1 Numerical Computation

We numerically search for equilibria that satisfy the conditions outlined above, and report the values of $\{V(\omega)\}_{\omega \in \Omega_N}$ therein, as well as the strategy profile (π) and the transition probability matrix (P^{π}) as needed. Our numerical procedure is to 'guess and verify' the equilibria, starting from the "maximal trading" strategy[32]. We then check whether this default strategy can sustain an equilibrium, if not, search alternative strategies, repeating the loop until a sustainable equilibrium strategy is found. Additional detail on this numerical procedure is provided in the Appendix. We start with $N = 3$ and raise the number of agents until we hit a computational barrier at around $N = 9$, which seems high enough to allow us to draw economic implications.

3.2 The Nash equilibria and State-contingent Values

Recall that any sample path $\{\Omega_t\}_{t=0}^{\infty}$ on a Markov process generated by strategies (π) –that satisfy optimality, feasibility and rational expectations– constitutes an equilibrium. However, it would be pointless to enumerate each sample path here. For the sake of concise presentation, we therefore concentrate on describing the properties of each state, in particular, the values attached to each state ($\{V(\omega)\}_{\{\omega \in \Omega_N\}}$), under the implicit understanding that each state (ω) belongs to a Markov chain generated by *equilibrium* strategies. This presentation will also be helpful in describing the incentives to coalesce in sections that follow.

3.2.1 An Illustrative Result: $V(\omega_i)$ for $N = 3; \beta = 0.96, u = 1, c = 0.3$

As an example[33], Figure 2 below depicts state-contingent values for $N = 3$ when $\beta = 0.96, u = 1, c = 0.3$. Each $V(\omega_i), i = 1, \ldots, 6$ on the left panel represents values that accrue to the agent conditional on being in each state, $\Omega_t = \{m_t, \Delta_t\}$. The vertical lines (bright red) are demarcations for *distributional* 'super-states'[34] (Δ_t), as can be read off the right hand panel as well. We first observe a few facts in this example which, in fact, also tends to hold in greater generality.

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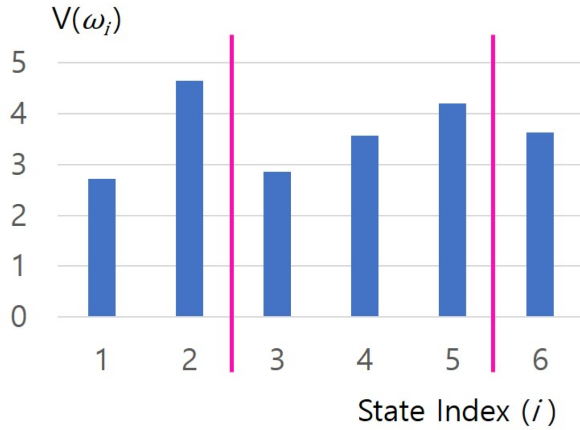
[32] This default strategy is motivated by the fact that $\beta < 1$, and that $u > c > 0$, whence it is reasonable to conjecture that it is beneficial to consume whenever agents have the chance to do so, rather than hold on to the money and wait for the next opportunity.

[33] Among the multitude of Nash equilibria discussed in 2.6, this example pertains to the "maximal trading" equilibria, which is the one conventionally discussed in the monetary economics literature.

[34] We use this terminology as an antonym of *sub*states. For example, both $\{(⓪, 0, 3)\}$ and $\{(0, 0, ③)\}$ share the same distributional super-state; $(0, 0, 3)$.



Index (i)	Notation
1	(0,0,3)
2	(0,0,3)
3	(0,1,2)
4	(0,1,2)
5	(0,1,2)
6	(1,1,1)

Figure 2: State-contingent values for $N = 3; \beta = 0.96, u = 1, c = 0.3$

Note that $V(\omega_1)$ and $V(\omega_3)$ both represent states where the agent holds zero stock of money, although they are in different distributional super-states. Specifically, $V(\omega_1)$ is in a more 'concentrated' distributional super-state; $(0, 0, 3)$, than $V(\omega_3)$ which resides in a relatively less concentrated distributional super-state; $(0, 1, 2)$. Similarly, $V(\omega_4)$ and $V(\omega_6)$ both represent states where agents hold one unit of money, albeit $V(\omega_4)$ is in a more concentrated distributional super-state than $V(\omega_6)$. First, we observe that

$$3.57 = V\big(\{(0, 1, 2)\}\big) = V(\omega_4) \neq V(\omega_6) = V\big(\{(1, 1, 1)\}\big) = 3.64 \tag{5}$$

This explicitly shows that the values attached to holding *identical* amount of money –in this case, a single unit– is *not* even across distributional states. That is, contrary to the conventional treatment in the literature where the money stock (m_t) is commonly the primary (if not unique) state variable, this inequality justifies our approach to look at each distributional state separately.

Similar to (5), we also note that

$$2.72 = V\big(\{(0, 0, 3)\}\big) = V(\omega_1) \neq V(\omega_3) = V\big(\{(0, 1, 2)\}\big) = 2.86 > 0. \tag{6}$$

This confirms our earlier assertion that value of holding money depends on the distributional states. Furthermore, (5) and (6) both inform us that the value of money is higher when the distribution is more even, rather than concentrated. This is intuitive; the more concentrated the money distribution, the more likely it is for a monopolizing money-holder to meet a trading partner who is unable to consume due to lack of money in stock. In turn, the partner's inability to trade is detrimental to the money-holder as well, as this means a foregone opportunity to hold an *additional* unit of money, which could have been used for future consumption that gives positive net utility value (since $\beta^t u - c > 0$ for reasonably high values of β.)

We also observe from (6) that the value associated to every state –even with *zero* holdings of money– is still positive. This is also intuitive, since the value of participating in the money-exchange economy includes the almost sure possibility of being assigned a producer in the future and procuring a unit of money in exchange of production, thereby allowing the agent to consume. This establishes that the "participation constraint" is always met in this money exchange economy. And finally, we note that the largest cross-sectional difference in V, within a given distributional super-state, is $V(\omega_1) - V(\omega_2)$, which is when the money-distribution is most skewed; $(0, 0, 3)$. It is reasonable to conjecture that higher levels of distributional skewness tend to enlarge the incentive to coalesce, which we confirm in sections that follow.

3.2.2 The Equilibrium Strategy Profile

We continue with the example with $N = 3; \beta = 0.96, u = 1, c = 0.3$. As $V(\omega_i)$'s need to be supported by equilibrium strategies $\pi : \Omega_N \times \{\mathcal{T}_N(\omega_i)\}_{\omega_i \in \Omega_N} \longrightarrow \{1, 0\}$, we now graphically depict the π corresponding to the above example in Figure 3 below. The left (right) panel corresponds to the consumer/buyer's (producer/seller's) strategy profile. The yellow margins denote state indices; $i = 1, \ldots, 6$ for our example with $N = 3$. We encode the strategy of a trader as binary entries on the matrix; element (i, j) denotes the trading decision of a trader in state ω_i paired with a trader in state ω_j[35]. Note that some matrix entries represent non-states; states that can never occur in reality, by construction. The strategy profile is therefore restricted to the green entries. Figure 3 shows that in our current example, the trading strategy profile that supports the equilibrium is simply to "trade whenever possible," i.e, identically 1. However, this triviality does not always hold, as we see in the next example.

[35]Note that specifying both ω_i and ω_j, together with knowledge of consumer/producer assignment is tantamount to knowing '$\Omega_{t+\epsilon}$', hence allows for this graphical representation of $\pi : \Omega_N \times \{\mathcal{T}_N(\omega_i)\}_{\omega_i \in \Omega_N} \longrightarrow \{1, 0\}$.

Consumer's Strategy

	1	2	3	4	5	6
1	1	1	-	-	-	-
2	1	-	-	-	-	-
3	-	-	-	1	1	-
4	-	-	1	-	1	-
5	-	-	1	1	-	-
6	-	-	-	-	-	1

Producer's Strategy

	1	2	3	4	5	6
1	1	1	-	-	-	-
2	1	-	-	-	-	-
3	-	-	-	1	1	-
4	-	-	1	-	1	-
5	-	-	1	1	-	-
6	-	-	-	-	-	1

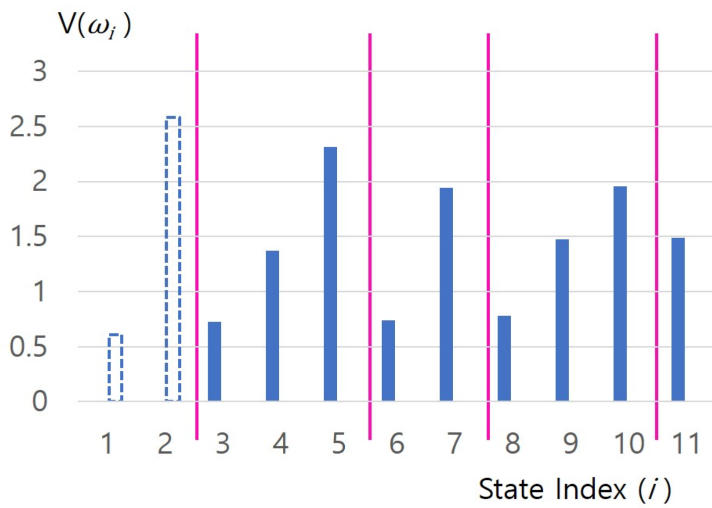
Figure 3: Equilibrium strategy profile (π) when $N = 3; \beta = 0.96, u = 1, c = 0.3$

3.2.3 A Different Example: $N = 4; \beta = 0.85, u = 1, c = 0.3$

We now provide another example, with $N = 4$. The previous example with $N = 3$ was 'standard' in the sense that all states –including the most skewed money distribution; $(0, 0, 3)$– are recurrent (i.e., almost surely revisited, infinitely often). On the other hand, the new example shows that in some cases, the optimal equilibrium strategy endogenously precludes such high concentration in the money distribution. Figure 4 is nearly identical to Figure 2, except it depicts $\{V(\omega_i)\}_{\omega_i \in \Omega_N}$ for $N = 4$ with lower β ($\beta = 0.85$). The intuitions we have gathered from equations (5) and (6) carry over to this example as well. However, unlike in Figure 2, states ω_1 and ω_2 do not arise in equilibrium, hence these states are represented as hollow bars in Figure 4.[36] Note that these unattainable states are associated with the most skewed money distribution super-state, $(0, 0, 0, 4)$.

The reason why the most skewed super-state is never attained is evident from the strategy profile, which we represent in Figure 5. While the consumer/buyer always trades, the producer/seller in state 5, i.e. $(0, 0, 1, 3)$, refuses to trade with a consumer in state 4, i.e. $(0, 0, 1, 3)$, preventing migration to states $(0, 0, 0, 4)$ and $(0, 0, 0, 4)$. This behavior is intuitive, and is closely related to our previous interpretation of equation (6). The producer contemplating a sale in $(0, 0, 1, 3)$ weighs the benefit of owning an extra unit of money against the costs as (s)he decides upon a trading strategy. The obvious benefit is the expected utility from *future* consumption that this acquired unit of money facilitates, however this benefit is decreasing in β, as future utility is more heavily discounted. Meanwhile, the cost to produce and sell –other than the obvious $-c$– is that doing so

[36] To be completely precise, these states *can* be given as the initial point in the evolution of the Markov chain, and for this reason, $V(\omega_1), V(\omega_2)$ can technically be computed as is shown in Figure 4. But this is the only possibility where these states can arise: once the states evolve, states 1 and 2 are never revisited, i.e., they are *transient* states. That these states never otherwise arise can be shown rigorously by computing the stationary distribution of the transition probability P^π, whereby the states are assigned zero probabilities.



Index (i)	Notation
1	(0,0,0,4)
2	(0,0,0,4)
3	(0,0,1,3)
4	(0,0,1,3)
5	(0,0,1,3)
6	(0,0,2,2)
7	(0,0,2,2)
8	(0,1,1,2)
9	(0,1,1,2)
10	(0,1,1,2)
11	(1,1,1,1)

Figure 4: State-contingent values for $N = 4; \beta = 0.85, u = 1, c = 0.3$

would further skew the distribution to the point of monopolizing the money stock of the economy, undermining her very own chances of future trades as potential producer. In this example, β is deliberately set to be low enough to depress the benefit, to the point that the benefit is outweighed by cost.[37] Hence the producer in this state refuses to trade, as is marked by the red tile in the right-hand panel of Figure 5 (i.e., $\pi^P\big(3, 1, P, (0, 0, 1, 3)\big) = 0$).

We emphasize two key implications elucidated by this example. First, the optimal choice in this example reveals that the value of money exhibits a *network* effect; namely, money is more valuable if it is held more evenly across the economy because the even spread reduces search friction and fosters trade. This is evidenced by the agent's deliberate avoidance of a skewed money distribution, even at the expense of a potential loss of their own future consumption. Secondly and relatedly, this example reinforces our focus on exploring the effect of the *distributional state* on money holdings, over and above the bare amount of money held, as our example clearly shows that rational agents must take the distributional effects into consideration when they make trading decisions. Perhaps more realistically, the implications drawn from this example provide potentially useful recommendations to future designers of cryptocurrencies. While the design of the currently existing

[37]Some readers may find the value of β in our example unrealistically low ($\beta = 0.85$). Endogenous trade refusals do occur, however, at reasonable values of β given different values of u and c. To explore this, we fix u at $u = 1$ without loss of generality, and vary c from $c = 0.3$ to $c = 0.6$ in increments of 0.1. The "least upper bound" of β that induce trade refusals are 0.895, 0.925, 0.945, 0.955 for c values of 0.3, 0.4, 0.5, 0.6, respectively. This shows that when c is sufficiently close to u (for example, when $c = 0.6$ or higher), endogeneous trade refusals many occur with very realistic values of β (for example, $\beta = 0.95$ or higher).

Consumer's Strategy

	1	2	3	4	5	6	7	8	9	10	11
1	1	1	-	-	-	-	-	-	-	-	-
2	1	-	-	-	-	-	-	-	-	-	-
3	-	-	1	1	1	-	-	-	-	-	-
4	-	-	1	-	1	-	-	-	-	-	-
5	-	-	1	1	-	-	-	-	-	-	-
6	-	-	-	-	-	1	1	-	-	-	-
7	-	-	-	-	-	1	1	-	-	-	-
8	-	-	-	-	-	-	-	-	1	1	-
9	-	-	-	-	-	-	-	1	1	1	-
10	-	-	-	-	-	-	-	1	1	-	-
11	-	-	-	-	-	-	-	-	-	-	1

Producer's Strategy

	1	2	3	4	5	6	7	8	9	10	11
1	1	1	-	-	-	-	-	-	-	-	-
2	1	-	-	-	-	-	-	-	-	-	-
3	-	-	1	1	1	-	-	-	-	-	-
4	-	-	1	-	1	-	-	-	-	-	-
5	-	-	1	0	-	-	-	-	-	-	-
6	-	-	-	-	-	1	1	-	-	-	-
7	-	-	-	-	-	1	1	-	-	-	-
8	-	-	-	-	-	-	-	-	1	1	-
9	-	-	-	-	-	-	-	1	1	1	-
10	-	-	-	-	-	-	-	1	1	-	-
11	-	-	-	-	-	-	-	-	-	-	1

Figure 5: Equilibrium strategy profile (π) when $N = 4; \beta = 0.85, u = 1, c = 0.3$

cryptocurrencies seem to be keen on ensuring that its supply is disciplined, this has also naturally led to a skewed distribution of these cryptocurrencies, stemming predominantly from differences in mining capabilities. Our model provides an insight, and a challenge to cryptocurrencies: to increase the value of money in circulation, it is important to flatten the distribution of money, given the network effect. On the other hand, it is also important to ensure that the aggregate supply is reigned in so as to avoid inflating away its value. (See, for example, Ritter 1995 or Taub 1985 for the importance of curbing unfettered issuance.) Attaining these goals simultaneously may pose a realistic challenge.

3.3 A (Graphical) Summary and Next Steps

In this section, we explored the consequences of considering the effects of money distributions explicitly. In particular, we document a "network effect" of money. Consequently, the value of money –even if possessed in identical amounts– may differ depending on the aggregate money distribution state of the economy. Namely, the value is higher the more uniform the distribution. Equilibrium strategies reflect this, by voluntarily refusing trades that may result in skewing the distribution excessively.

Figure 6 succinctly summarizes this through a computation result with $N = 8$. The upper panel plots state-values ($V(\omega_i)$'s) along two state-related dimensions. G orders states by the level of inequality of money distributions as measured by the Gini coefficient, with larger values of G corresponding to more equal distributions (lower Gini coefficients), and m is the units of money in possession in each state. For example, the dark blue bars all pertain to states where agents hold $m = 0$, and the yellow bar depicts the value of the state with $m = 8$.[38] The lower panels provide front views from two angles. The lower-right panel clearly shows that there is a network effect: for any series with a given m (i.e., same color), state values tends to grow as Gini coefficients are lowered (more equal distribution). The lower-left panel show values contingent on m. While state-values naturally increase with the amount of money possessed, the graph shows that the increment is not completely linear, but rather concave. This feature is another manifestation of the network effect of money. An increment of m is increasingly less appreciated as m begin to rise to levels that imply unequal distributions of money; if $m \approx N$ in the extreme, very few agents monopolize the entire stock of money in the economy, which is detrimental to the value of money in circulation as per the network effect.

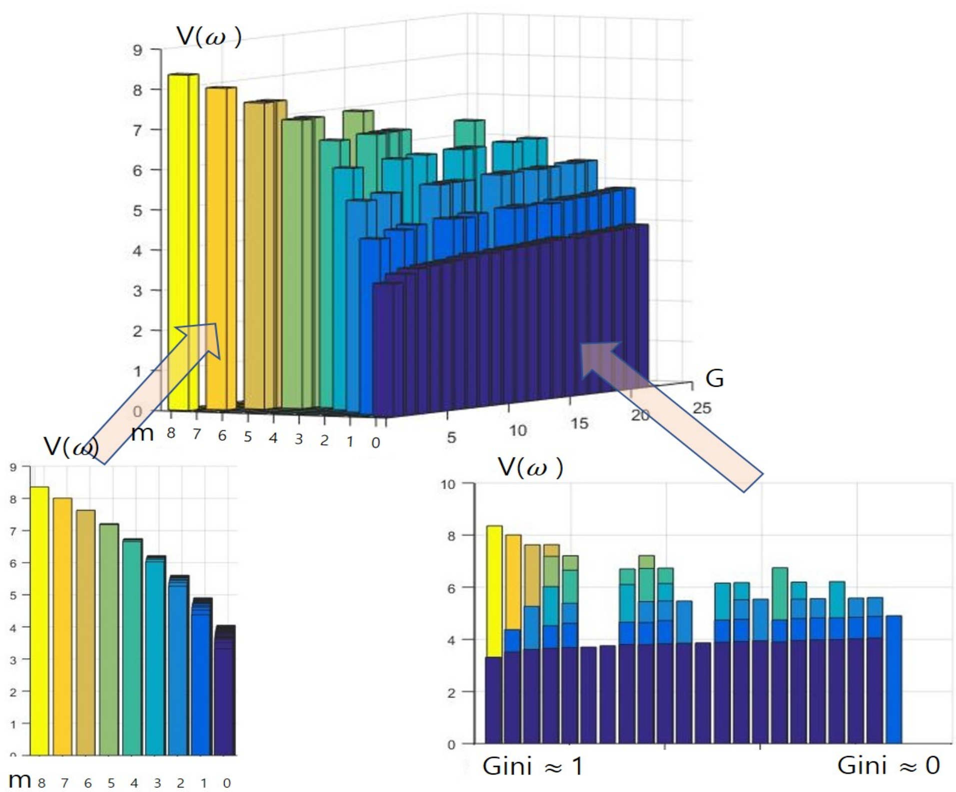


Figure 6: The “Network Effect” of Money: $N = 8; \beta = 0.96, u = 1, c = 0.3$

[38]Note that the number of bars decrease with m because higher m implies fewer distributional super-states. For example, there are numerous states where agents hold $m = 0$: $(0, \ldots, 0, 8)$, $(0, \ldots, 0, 1, 7)$, $(0, \ldots, 0, 1, 1, 6)$, $(0, \ldots, 0, 1, 1, 1, 5)$, $\ldots$, $(0, 0, 0, 0, 2, 2, 2, 2)$, etc., with 21 in total. However there is a single distributional super-state where money is monopolized: $(0, \ldots, 0, 8)$.

As noted above, the distributional effect we point out already provides some implication for cryptocurrencies and their design. Meanwhile, given the network effect of money, one may also conjecture that if the distribution of the current money in circulation is unequal enough, the inequality may incentivize a group of people to form a coalition, so as to repudiate the currency and deviate to a new equilibrium where the money distribution is more uniform. We explore this possibility and its consequences in the section that follows.

4 The "Joint Deviation" Model

Cryptocurrencies are generally not legal tender, hence it is realistically possible to see privately-made decisions to retreat those in existence and start anew. In this context, we ask: "Is the Nash equilibrium stable/sustainable?" To do this, we expand the baseline model ("Nash model") to incorporate the strategic choice to form coalitions and jointly deviate, whose precise meaning we will explain in the description that follows. The joint deviation we consider can be thought of as a simplification of those (Coalition-Proof Nash equilibrium) considered by Bernheim et. al. (1987), while retaining its core spirit.

4.1 Joint Deviation by Voting: the Mechanism and Timeline

We tweak the setup of the baseline model and give the agents an opportunity to vote, once their trades are settled. The voting rule dictates that upon reaching a pre-announced quorum, the incumbent money in circulation is discarded and its purchasing power repudiated, after which a new money is issued and distributed equally among all agents in the economy in the period that follows. Also, for the sake of realism[39], we introduce a small but positive re-issuance cost, $k > 0$.[40]

More formally, we add $\theta : \Omega_N \longrightarrow \{0, 1\}$ to the original strategy space (π), where $\theta = 0$ represents a 'no' and $\theta = 1$ represents a 'yes'. Let $Q \leq N$ denote the quorum required to make the decision to "repudiate and re-issue". If Q (or more) agents vote $\{1\}$, the currency in circulation is discarded and a new currency is introduced, distributed evenly across every agent within the

[39] For paper money, this cost includes real and material costs such as printing and transporting new money, replacing ATM machines, etc. For cryptocurrencies this may represent the cost to administer the vote, advertise and announce the results, establish new platforms, the technological efforts necessary to ensure that the transition is seamless.

[40] Alternatively, we may have imposed $k = 0$ instead, however $k > 0$ only stacks the odds against our prior that the "Nash equilibrium" must be susceptible to joint deviations, as it will make the joint deviation more costly.

economy, including those who voted $\{0\}$. To simplify matters by disentangling π and θ decisions as much as possible[41], we assume that the execution of θ takes place *after* the implementation π (the time of which, recall, we denoted by '$t+\epsilon$'), but *before* $t+1$. Let '$t+\frac{1}{2}$' denote the time at which θ is executed, where as assumed, $t+\epsilon < t+\frac{1}{2} < t+1$. Let $\{m_{t+1}^{-\theta}, \Delta_{t+1}^{-\theta}\}$ denote the state that would arise at $t+1$ upon execution of π *only*, that is, without the subsequent execution of θ. Our assumption is that $\{m_{t+1}^{-\theta}, \Delta_{t+1}^{-\theta}\}$ is known at '$t+\frac{1}{2}$' when the voting decisions are made: i.e., $\Omega_{t+\frac{1}{2}} = \{m_{t+1}^{-\theta}, \Delta_{t+1}^{-\theta}\}$. Simply put, the assumption is that when agents make the voting decisions (θ), they have a clear understanding of what state they will walk into –as the collective consequence of their trade strategies (π)– if it had not been for the voting results. Once the voting decisions (θ) are made at $t+\frac{1}{2}$, its outcome is implemented at $t+1$ by repudiating the old and re-issuing new money, conditional on passing the pre-determined quorum (Q). If the vote is to repudiate and re-issue, then all agents are levied an egalitarian tax of amount kV, $0 < k < 1$, where V is the value of money in the repudiated and re-issued state. The rest is analogous to the setup of the "Nash model". The set of strategies (π, θ) jointly determine the $t+1$-period state ($\Omega_{t+1} = \{m_{t+1}, \Delta_{t+1}\}$) in any generic time interval $[t, t+1]$, as well as the equilibrium values $\{V(\omega_i)\}_{\omega_i \in \Omega_N}$ therein. And as in the "Nash model", those who find it unusual that agents must use $t+\frac{1}{2}$-information ($\Omega_{t+\frac{1}{2}}$) for t-period decisions can think of θ as a contingency plan set in place at time t, mechanically executed as information unfolds. Figure 7 below summarizes the timeline and information structure augmented to incorporate voting strategies θ.

4.2 The Equilibrium with Joint Deviation: "Q-Nash" Equilibrium

Since the voting mechanism is now added, we need to define a (slightly) different equilibrium concept. We first modify the agent's objective (1) conformably. Agents here seek to maximize:

$$V_t(\Omega_t) = \mathbb{E}_{\pi,\theta}\left[\sum_{s=t}^{\infty} \beta^{s-t} \Phi_{\pi}^{-\theta}\left(\pi^j\left(\Omega_{s+\epsilon}, \theta(\Omega_{s+\frac{1}{2}})\right) \cdot \mathbb{1}_F(\Omega_{s+\epsilon})\middle|\Omega_t\right)\right], \tag{7}$$

[41]That is, to obviate the complications associated with forecasting the next period state from matching result (τ) as in the execution of π, we simply assume that agent make voting decisions *after* they are informed of the state that will arrive in the next period. While τ and the forecasting process must be taken into account to lay down the *incentive compatibility* conditions (because they jointly determine the states in the next period), this is not necessary in the execution of θ because it is entirely plausible to assume that the voting decisions are based on the states that are due to arise in the absence of voting decisions and results.

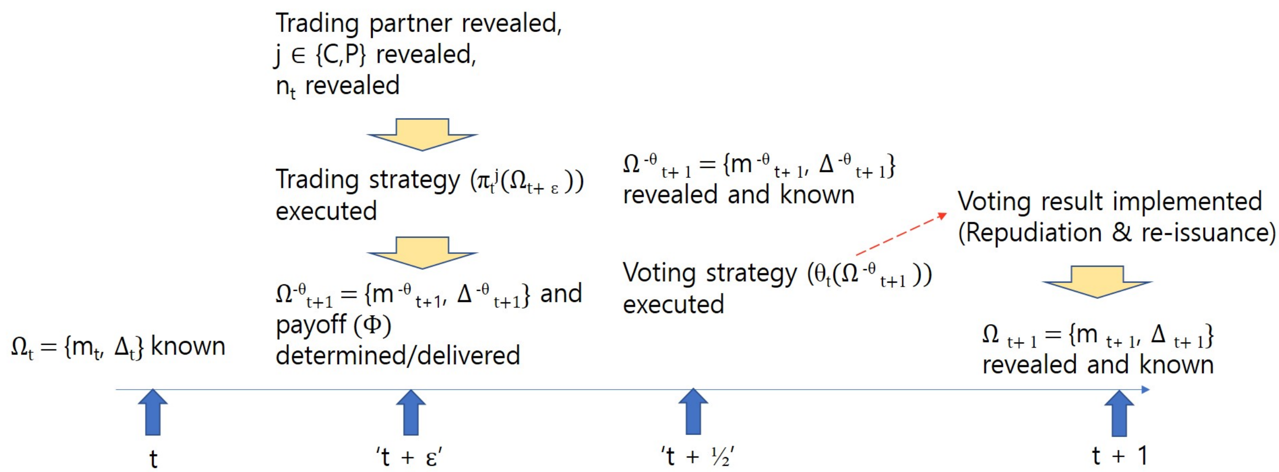


Figure 7: A representative time node $[t, t+1)$, with voting

where the payoffs $\Phi_\pi^{-\theta}[\cdot]$[42] and the conditional expectations $\mathbb{E}_{\pi,\theta}[\cdot]$[43] now clearly depend on θ, as well as π. By construction, we can index a voting rule by its quorum, Q. The equilibrium for a given Q, which we call the (symmetric)"*Q-Nash* equilibrium," is defined as: a sequence of $\{\Omega_t\}_0^\infty$, its probability transition matrix P^π, and (symmetric) strategy profiles $\pi : \Omega_N \times \{\mathcal{T}_N(\omega_i)\}_{\omega_i \in \Omega_N} \longrightarrow \{1, 0\}$ *and* $\theta : \Omega_N \longrightarrow \{0, 1\}$, such that (*i*) [*Deviation by Voting*]: new money is issued if and only if Q or more agents vote in its favor, and each agent votes in favor if and only if such action generates strictly higher expected utility for the individual. Additionally, the Q-Nash equilibrium must also satisfy the following three requirements that carry over from the baseline model: (*ii*) [*Optimality*] Each agent chooses her strategy profiles (π, θ) to maximize equation (7) at each t, subject to the equilibrium strategy profiles of other agents, the given probability transition matrix $P^{\pi,\theta}$ and nature's trading pair assignment rules. (*iii*) [*Feasiblity*] The chosen trading strategy (π) is executed if and only if it is feasible. (*iv*) [*Rational Expectations*] The probability transition matrix $P^{\pi,\theta}$ and the evolution of $\{\Omega_t\}_t^\infty$ is consistent with the equilibrium strategy profiles (π, θ) and nature's trading pair assignment rules.

[42] $\Phi_\pi^{-\theta}$ denotes the payoffs that are defrayed *prior* to the execution of θ, as per our timeline and information structure. Note also, that the t-period payoffs $\Phi_\pi^{-\theta}$ continue to depend directly on '$t+\epsilon$'-period trading strategy (π) as before, but π themselves depend on the '$t+\frac{1}{2}$'-period voting strategy (θ), hence generating an indirect dependence on θ as well. See Figure 7 for a summary of the timeline structure.

[43] The conditional expectations need to be modified because the transition probabilities depend jointly on strategies π *and* θ.

The corresponding Bellman representation is obtained as in (2) - (3), *mutatis mutandis*:

$$V(\omega) = \max_{\pi,\theta} \mathbb{E}_{\pi,\theta}[\Phi_\pi^{-\theta} + \beta \cdot V(\omega')|\omega], \;\; \omega \in \Omega_N, \tag{8}$$

and similarly, a more explicit expression is given by:

$$V(\omega_i) = \sum_{\omega_j \in \left\{\mathbb{T}_\tau^{\pi,\theta}(\omega_i)\right\}_{\tau \in \mathcal{T}_N(\omega_i)}} P_{i,j}^{\pi,\theta} \cdot \Big(\underbrace{u \cdot \mathbb{1}_{\{m^{-\theta}(\omega_i) - m^{-\theta}(\omega_j) = 1\}} - c \cdot \mathbb{1}_{\{m^{-\theta}(\omega_i) - m^{-\theta}(\omega_j) = -1\}}}_{\Phi_\pi^{-\theta}} + \beta \cdot \underbrace{V(\omega_j)}_{V(\omega')} \Big),$$
$$\omega_i \in \Omega_N, \tag{9}$$

where the equilibrium strategies must now satisfy the incentive compatibility conditions pertaining to θ:

$$(\forall \omega \in \Omega_N), \quad \hat{\theta}(\omega) = 1 \quad \text{if and only if} \quad V(\omega')\Big|_{\hat{\theta}(\omega)=1} > V(\omega')\Big|_{\hat{\theta}(\omega)=0}, \tag{10}$$

as well as the incentive compatibility conditions pertaining to π that carry over from (4):

$$(\forall \tau \in \mathcal{T}_N(\omega) \text{ and } \forall \omega \in \Omega_N), \quad \hat{\pi}(\omega,\tau) = 1$$
$$\text{if and only if} \quad \big(\Phi_{\hat{\pi},\theta} + \beta \mathbb{E}_{\hat{\pi},\theta}[V(\omega')|\omega,\tau]\big)\Big|_{\hat{\pi}(\omega,\tau)=1} > \big(\Phi_{\hat{\pi},\theta} + \beta \mathbb{E}_{\hat{\pi},\theta}[V(\omega')|\omega,\tau]\big)\Big|_{\hat{\pi}(\omega,\tau)=0}. \tag{11}$$

4.3 The Incentive to Deviate and the "Voting-proof" Nash Equilibrium

Recall that the "Nash" equilibrium (NE) pertains to a model that does *not* allow for joint deviation, whereas "Q-Nash" equilibrium (Q-NE) belongs to one where joint deviation is allowed. It is natural to ask how the two compare. We formally show that the "Q-Nash equilibrium" is meaningfully different from the "Nash" equilibrium, and also elucidate this with an example. The non-trivial difference between the two equilibrium concepts implies that the traditional "Nash" monetary equilibrium –without due considerations given to distributional effects– may need to be refined, in the sense that they would suffer from the incentive to deviate away by forming coalitions, in particular, by voting it away.

4.3.1 Voting-Proof Nash Equilibrium

A direct comparison of "Nash" and "Q-Nash" equilibria is not feasible in general because, after all, they arise from different models. We therefore start by specifying a subset – of "Q-Nash" equilibria – that *does* allow for direct comparison with its "Nash" counterpart.

Definition 1 (Voting-Proof Nash Equilibrium)**.** A Q-Nash equilibrium is a *voting-proof Nash equilibrium* (VPNE) if, on every equilibrium path, $\theta = 0$ holds.

VPNE is a subset of Q-NE where joint deviation is never exercised ($\theta \equiv 0$) in equilibrium. Consequently, this raises some hope that the equilibrium strategy profile is directly comparable with those in NE, even though they belong to different games (models). In the next subsection, we show that the set of VPNE is "properly contained" in the set of NE.

4.3.2 VPNE is a Non-trivial Refinement of NE

Lemma 2. *Let $\mathbb{V}^*$ and $\mathbb{N}^*$ denote the set of all VPNE and NE, respectively. Then $\exists \varphi : \mathbb{V}^* \hookrightarrow \mathbb{N}^*$, an "embedding" such that:*
(i) φ is an injection (i.e., one-to-one mapping), and
(ii) φ preserves all trading strategies (π) and transition probabilities ($P^{\pi} = P^{\pi,\theta}$).

Lemma 2-(i) asserts that for every VPNE, there is one (and only one) NE which it corresponds to, while Lemma 2-(ii) informs us that the equilibrium allocations in VPNE ($\mathbb{V}^*$) and its corresponding counterpart ($\varphi(\mathbb{V}^*) \subset \mathbb{N}^*$) must be identical. Thus, taken together, Lemma 2-(i) and (ii) imply that $\mathbb{V}^*$ can be "embedded" into $\mathbb{N}^*$ effortlessly (via φ) as they are identical in every economically meaningful aspect.[44] This is graphically depicted in Figure 8 below, which clearly shows that VPNE is "contained" in NE as a structure-preserving set. Note that this containment should not be obvious, as the (off-equilibrium) possibility to jointly deviate *may* induce trading strategies that are different from those of the "Nash" model, even when the vote is never exercised in equilibrium ($\theta = 0$). Lemma 2 dictates that this is not the case. The following Lemma further informs us that the containment is, in fact, "proper".

Lemma 3. *Let $\mathbb{Q}^*$ denote the set of all Q-Nash equilibria for any given $Q < N$. Then $\exists n \in \mathbb{N}^*$ whose equilibrium allocation is not attained in $\mathbb{Q}^*$.*

Lemma 3 effectively states that not every NE allocation can be sustained by a Q-NE. Given that VPNE is a Q-NE that *does* sustain its NE counterpart (Lemma 2), this implies that at least some elements in NE (e.g., $n \in \mathbb{N}^*$) do not correspond to a matching VPNE. Therefore, these NE would not have been "voting-proof" equilibria[45], whereas VPNE, on the other hand, *are* voting-proof

[44] VPNE ($\mathbb{V}^*$) can thus be viewed as a "subspace" of NE ($\mathbb{N}^*$): $\mathbb{V}^* \subset \mathbb{N}^*$.

[45] In the sense that they would have been eliminated by vote ($\theta = 1$) if agents were given the opportunity to repudiate and re-issue. Note also that this, in turn, implies that $\mathbb{V}^* \subsetneq \mathbb{N}^*$.

Q-NE that can be identified (via φ) with elements in $\mathbb{N}^*$. Hence it follows that VPNE constitutes a non-trivial refinement of NE. We depict this refinement graphically in Figure 8 as a proper subset of $\mathbb{N}^*$.

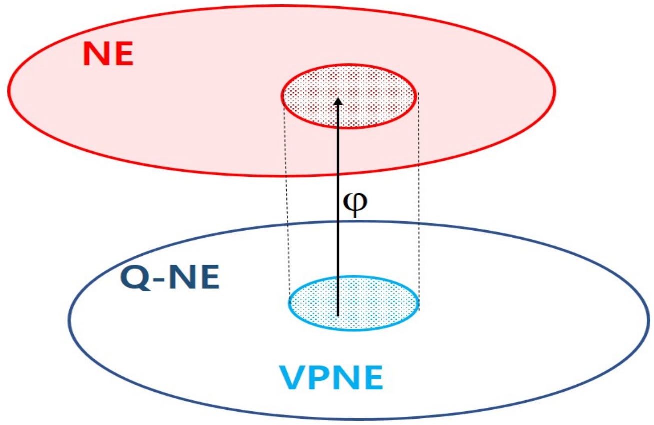


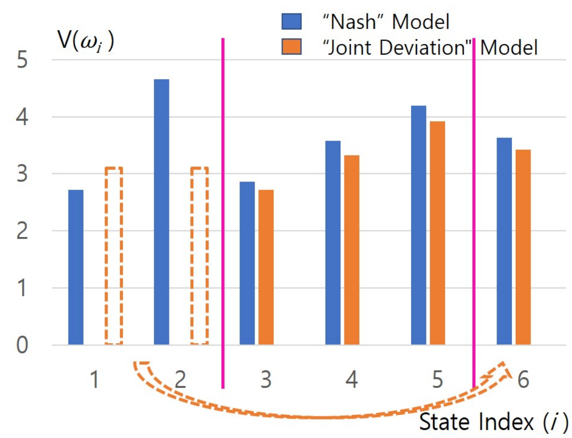
Figure 8: NE, Q-NE, and VPNE

4.3.3 Example: An Earlier Result Revisited

We illustrate the refinement through an example. Recall the earlier example of a "Nash" model in 3.2.1 with $N = 3; \beta = 0.96, u = 1, c = 0.3$. For ease of comparison, we recycle the parameters in 3.2.1 and further let $Q = N - 1 = 2$ and $k = 0.05$ in the "joint deviation" model. As before, Figure 9 displays $V(\omega_i)$'s ($\omega_i \in \Omega_N$) arising from the equilibrium outcomes. The blue bars represent values associated to a NE[46], whereas $V(\omega_i)$'s in the corresponding Q-NE are represented by orange bars. Naturally, the blue bars are identical to those in Figure 2. The orange bars that represent Q-NE, however, are no longer identical; for example, states 1 and 2 are non-states, and the $V(\omega_i)$'s in the remaining states are also lower than their counterparts in the NE. This difference – the reason for which we will shortly explain – is a clear justification of our initial motivation and claim; it unequivocally shows that the incentive to jointly deviate can indeed unravel a NE, which we believe is an important aspect that has hitherto been overlooked in the analysis of monetary equilibrium, especially so in the context of the rapid developments in technology that are conducive to such collective actions.[47]

[46]There can be multiple NE, and the blue bars represent an equilibrium where trades are maximally facilitated.

[47]In this Example, we fix k at $k = 0.05$. Naturally, increasing k disincentivizes repudiation. Numerically, the threshold –above which repudiation ceases to occur– in this Example is in between $k = 0.22$ and $k = 0.23$.



Index (i)	Notation
1	(⓪,0,3)
2	(0,0,③)
3	(⓪,1,2)
4	(0,①,2)
5	(0,1,②)
6	(①,1,1)

Figure 9: State-contingent values $V(\omega_i)$ in NE and Q-NE

To understand the mechanism of the unraveling, we first look at the voting strategies (θ). Recalling that $(0, 0, 3)$ represents the most unequal money distribution super-state when $N = 3$, it is reasonable to intuit that those who hold $m = 0$ in this super-state would seek to subvert by voting to repudiate ($\theta = 1$). Moreover, they would execute this strategy knowing that the quorum ($Q = 2$) will be attained, since there are two such agents. The intuition is in fact valid, because the value of repudiating – and hence receiving the value associated to (①, 1, 1) next period – is still larger than complying with the incumbent monetary equilibrium (⓪, 0, 3), even after discounting for time preference (β) and issuance costs (k). This can be seen on Figure 9, by comparing the relative heights of the blue bar in state ω_1 against the orange bar in state ω_6. Note that $(0, 0, 3)$ is the only super-state where quorum will be attained. In super-state $(0, 1, 2)$, only the agent in (⓪, 1, 2) will choose '$\theta = 1$' – and those in (0, ①, 2) or (0, 1, ②) will not, as can also be seen by comparing the blue bars in states ω_4 and ω_5 against the orange bar in state ω_6 – hence the vote would fall short of $Q = 2$. Similarly, every agent chooses '$\theta = 0$' in super-state $(1, 1, 1)$. In sum, the voting strategy (θ) dictates that the equilibrium will unravel to state ω_6 whenever states ω_1 and ω_2 are reached, as expressed by the orange bowed-arrow in Figure 9.

Given this voting strategy (θ), Figure 10 describes the trading strategies (π) of the Q-NE. Those marked as red refer to decisions where trade is refused ($\pi = 0$), as opposed to the previous example in 3.2.1 where every trade was endorsed. The trade refusals ($\theta = 0$) marked as light red do not pass the feasibility condition, and hence do not affect the equilibrium allocation.[48] However, the

[48]For example, that $\pi^P(\{0, 0, P,$ (⓪$, 0, 3)\}) = 0$ is not material in terms of determining equilibrium allocations,

refusal marked in dark red does affect allocations. That the producer (seller) in state ω_5, $(0, 1, 2)$, refuses to sell to a consumer (buyer) in state ω_4, $(0, 1, 2)$, effectively blocks off the state transition from $(0, 1, 2) \to (0, 0, 3)$, thereby isolating states ω_1 and ω_2 to be non-states, expressed as hollow dashed bars. Intuitively, this is because the agent in state $(0, 1, 2)$ will find it unappealing to expend the production cost (c), only to transition to state $(0, 0, 3)$ which (s)he knows will unravel to $(1, 1, 1)$. Furthermore, since the isolation of states ω_1 and ω_2 is a consequence of *missed* trading opportunities, it results in lowering values in the remaining states. This can be seen by comparing the orange and blue values in states $\omega_3 \cdots, \omega_6$ of Figure 9.

Consumer's Strategy

	1	2	3	4	5	6
1	0	0	-	-	-	-
2	1	-	-	-	-	-
3	-	-	-	0	0	-
4	-	-	1	-	1	-
5	-	-	1	1	-	-
6	-	-	-	-	-	1

Producer's Strategy

	1	2	3	4	5	6
1	0	1	-	-	-	-
2	0	-	-	-	-	-
3	-	-	-	1	1	-
4	-	-	0	-	1	-
5	-	-	0	0	-	-
6	-	-	-	-	-	1

Figure 10: Strategy profile (π) for Q-NE $(N = 3; \beta = 0.96, u = 1, c = 0.3, Q = 2, k = 0.05)$

Taken together, our analysis of (θ, π) reveals that the economy never reaches super-state $(0, 0, 3)$. Consequently, '$\theta = 1$' remains an off-equilibrium strategy, and the economy transitions within the remaining states $\omega_3, \cdots, \omega_6$ where the voting right is never exercised on equilibrium paths. It then follows from Definition 1, that the equilibrium represented by the orange bars (Figure 9) and the strategy matrices (Figure 10) is in fact a VPNE. Finally, this allows us to conclude, from Lemma 2, that the (solid) orange bars represent the NE which sustains the allocation of this VPNE: i.e., it is φ(VPNE).

4.4 Consequences of Lowering the Quorum (Q)

Intuitively, relaxing the quorum (Q) makes joint deviation easier. For example, in the most stringent case of $Q = N$, votes must be unanimous to deviate. Here, deviation is impossible even in the most unequally distributed super-state, $(0, 0, \ldots, 0, N)$, since the agent in state $(0, 0, \ldots, 0, N)$ will vote against a foreseeable drop from $m = N$ to $m = 1$. In the other extreme, relaxing the

since the buyer owns no stock of money $(n = 0)$, hence the trade was not feasible to begin with.

quorum down to $Q = 1$ would invariably devolve the equilibrium to an autarky because the smallest inequality could be voted away, and thereby unravel trade prospects. Figure 11 gives an example that validates this intuition with $N = 7$.

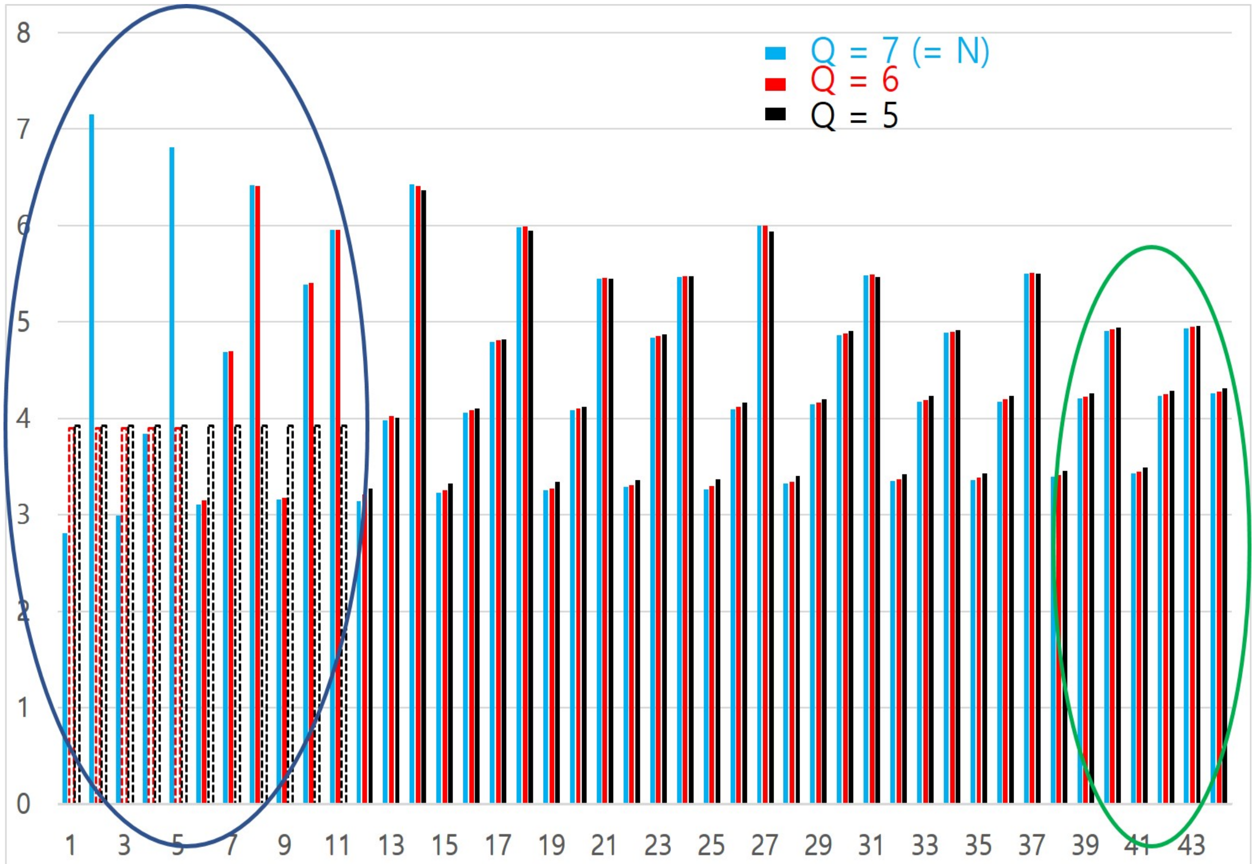


Figure 11: Relaxing the Quorum

As in Figure 9, the hollow dashed bars in Figure 11 denote states where the votes would have attained Q, even though these states are never reached on any equilibrium path because agents anticipate the voting outcome and trading strategies (π) preclude such a state (trade refusals). The circle on the left highlights these states that are voted away. As predicted, decreasing Q from N to $N-1$ and $N-2$ sequentially enlarges the states that are voted away: when $Q = 7$, no state is voted away; when $Q = 6$, states $\omega_1, \cdots, \omega_5$ are voted away; and when $Q = 5$, states $\omega_1, \cdots, \omega_{11}$ are voted away.

A natural question to ask is: when the quorum (Q) is lowered below N, who gains, who loses, and how does it affect the average (unconditional) value of money in the economy? Clearly, the high-m agents in the less equal distributions –for instance, $(0,\ldots,0,⑦)$ or $(0,\ldots,1,⑥)$– would lose, and low-m agents therein would gain as it becomes incrementally easier to repudiate. However, these highly uneven ('low entropy') distributions are probabilistically less likely to arise compared to the more evenly distributed states, and therefore, to understand how lowering the quorum (Q) impacts unconditional value, it is important to understand its impact on the more even distributions.

To this end, recall that the 'network effect' of money implies an unequivocal preference for an egalitarian distribution, and therefore, lowering Q offers a clear benefit: it prevents the economy from falling into a money distribution that is excessively unequal[49]. This benefit is also visually clear in Figure 11, for example, in the area marked by the right circle, where values in each state are shown to rise as Q is lowered. However, as was explained in 4.3.3, this benefit comes at the expense of trade refusals which undermine the value of money, as was indeed the case in 4.3.3 with $N = 3$ (Figure 9). And naturally, this cost tends to expand as we lower the Q vis-a-vis N, because lowered Q enlarges the non-states, which in turn, further deteriorates trade through trade refusals. In the current Example with $N = 7$ however, the benefit of averting excessive inequality still outweighs the cost, at least up to $Q = 5$, whereas in 4.3.3 with $N = 3$, $Q = 2$ was already low enough so that the benefit was overwhelmed by the costs. A pattern emerges: when Q is high ($Q \approx N$), relaxing the quorum is value-increasing for the economy[50]. The other extreme ($Q = 1$) is clearly value-destructive as it devolves into autarky. Hence, there must exist an "optimal-Q" (Q^*) in the range $1 < Q^* < N$ that maximizes the average value of money in the system. In particular, this means that a Q^*-NE is welfare-enhancing compared to NE (since $Q^* < N$), insofar as it helps avert a distribution where a few agents monopolize the entire supply of money. This sends a clear message to policy-makers and cryptocurrency designers alike: an excessively unequal money distribution is harmful to its value, and one way to guard against it may be to allow for value repudiation, with the caveat that it must be exercised under stringent conditions.

We end by formalizing the result seen in the current Example. Fixing all parameters other than Q, let VPNE_Q denote the VPNE with quorum Q, and let $\mathcal{Q}_{VI}$ denote the set of Q where decreasing

[49] Note that once an economy falls into such states, it typically takes time to dissolve into an evenly distributed state, whereas the voting alternative offers a 'quick way out'.

[50] These values can be formally computed from the stationary distribution of the Markov chain by taking its inner product with $V(\omega_i)$'s. Computations show that the average values are: 4.04 when $Q = 7$, 4.07 when $Q = 6$ and 4.14 when $Q = 5$, clearly increasing as Q is relaxed.

Q is value-increasing: i.e., $\mathcal{Q}_{VI}$ is such that for any $Q < Q' \in \mathcal{Q}_{VI}$, $V(\omega_N)_Q > V(\omega_N)_{Q'}$ holds.[51]

Lemma 4. *For any $Q' < Q$ in $\mathcal{Q}_{VI}$, $VNPE_{Q'} \subsetneq VNPE_Q$*

In line with observations made from Figure 11, Lemma 4 simply states that VPNEs increase *monotonically* with $Q \in \mathcal{Q}_{VI}$, since relaxing the quorum enlarges the states that are voted away (VPNE^c), and consequently, fewer and fewer states are sustained as VPNE[52].

5 Conclusion

We develop a search model to study monetary equilibrium under an environment where the distribution of money is allowed to change over time within finite states, reflecting transactions among finite number of agents.[53] Departing from stationary distributions elucidates an intuitive, yet undocumented fact; that the value of money depends on its cross-sectional distribution. That is, the value of a given unit of money is not identical across all possible distributional states, since the distribution affects search friction and hence the likelihood of successful trades. After establishing that there exists at least one non-trivial "Nash" monetary equilibrium under reasonable parameter values regarding consumption, production cost and discounting, we find a network effect in the value of money. We then allow agents to form a coalition to "repudiate and re-issue," and find that the incentive to jointly deviate is large when the distribution of money is concentrated. Accordingly, we suggest a "coalition-proof" refinement of the "Nash" monetary equilibria in the spirit of Bernheim et al (1987), which we call the "voting-proof Nash equilibrium". Our analysis is designed to be especially relevant in the context of cryptocurrencies, where information technology has significantly lowered the entry barrier associated to private issuance. With implications drawn from the network effect, our model also sheds some light on the on-going discussions on central bank digital currency (CBDC), insofar as CBDC –as a legal tender– is expected to be more evenly distributed relative to privately issued cryptocurrencies.

[51] $V(\omega_N)_Q$ denotes the value accrued to the state ω_N (the most egalitarian state) under the quorum Q.

[52] While we know that VPNE must increase with Q considering the fact that VPNE $= \mathbb{Q}^*$ when $Q = N$ and VPNE $= \emptyset$ when $Q = 1$, this Lemma specifies a region where this relation must be *monotonic*.

[53] It was suggested to us that it may be desirable to explore the consequences of increasing N, as it may 'stabilize' the apparent non-stationarity towards an asymptotic stationarity as the agents become atomistic. While we recognize the computational limitation ($N \approx 10$) of our model, our results with finite number of elements –where the equilibria are infallibly non-stationary– befits our purpose, which is primarily to explore the value and strategic interactions that arise when the distribution is allowed to vary over time.

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Appendix

Lemma 5 (Number of States). *For any $N \geq 3$, $n(\Omega_N) = \sum_{k=0}^{N} p(k) - 1$, where $p(k)$ denotes the number of* integer partitions *of k.*

Proof. Suppose that one individual, say agent i, has N unit of money, then there is only one way to distribute 0 unit of money to the remaining $N-1$ agents. Generally, if agent i has $N-k, k < N$ units of money then there are $p(k)$ ways to distribute k units of money to remaining $N-1$ agents. In a case with $k = N$, that is if agent i has 0 unit of money then there are $p(N) - 1$ ways to distribute N units to the remaining $N-1$ agents.[54] □

Proof of Lemma 1

Proof. Since (4) holds by assumption with $\pi \equiv 1$ under "reasonable" parameters, it suffices to check that (3) holds with $\pi \equiv 1$. Rewriting, this amounts to finding a v that satisfies the matrix equation: $v = r + \beta P v$, where P is the Markov transition matrix and r is the state-wise "expected payoff" vector, both *constants* under $\pi \equiv 1$. Since P is a Markov transition matrix, $I - \beta P$ has an inverse, hence v is uniquely determined as $v = (I - \beta P)^{-1} r$. □

Note that while a given "reasonable" parameter configuration sustains $\pi \equiv 1$ –in which case v is unique– it may also sustain other strategy profiles, hence this Lemma does not preclude multiple equilibria.

Proof of Lemma 2

Proof. (i) We fix some notations first. Since any $n \in \mathbb{N}^*$ (or $v \in \mathbb{V}^*$) uniquely defines on- and off-equilibrium states, we can define a partition of Ω_N, $\mathcal{P}(n) := \Omega_{eq}(n) \oplus \Omega_{-eq}(n)$ for any given n (or v). Also, $\mathbb{N}^*$ (or $\mathbb{V}^*$) defines an equivalence class on its strategy space Π (or $\Pi \times \Theta$), where each member of the class corresponds to a distinct[55] equilibrium, which we denote as $[\Pi]$ (or $[\Pi \times \Theta]$). And since any element in $[\Pi]$ (or $[\Pi \times \Theta]$) *uniquely* corresponds to an element in $\mathbb{N}$ (or $\mathbb{V}^*$) and vice versa, we may define mappings $\eta_N : [\Pi] \longrightarrow \mathbb{N}^*$ and $\eta_V : [\Pi \times \Theta] \longrightarrow \mathbb{V}^*$, both *bijections*[56] with respective inverses η_N^{-1} and η_V^{-1}, that represent these correspondences. By the partition $\mathcal{P}$, any $\eta_N^{-1}(n)$ (or $\eta_V^{-1}(v)$) can be decomposed into strategies on Ω_{eq} and Ω_{-eq}, which

[54] The case of distributing 1 unit of money to each and every agent must be excluded since this contradicts with the fact that agent i has 0 unit of money.

[55] That is, distinct up to on-equilibrium paths and strategies.

[56] This is essentially by virtue of being defined on equivalence classes.

we denote as $[\pi_{eq} \oplus \pi_{-eq}]$ (or $[(\pi_{eq}, \theta_{eq}) \oplus (\pi_{-eq}, \theta_{-eq})]$). Note the following fact that is easy to verify:

Fact 1 (i) : $[\pi] = [\pi'] \iff \pi_{eq} = \pi'_{eq}$ *and* $\mathcal{P}(\eta_N(\pi)) = \mathcal{P}(\eta_N(\pi'))$
(ii) $[(\pi, \theta)] = [(\pi', \theta')] \iff (\pi_{eq}, \theta_{eq}) = (\pi'_{eq}, \theta'_{eq})$ *and* $\mathcal{P}(\eta_V((\pi, \theta))) = \mathcal{P}(\eta_V((\pi', \theta')))$

Using Fact 1, one can also show the following.[57]:

Fact 2 (i) : $[\pi_{eq} \oplus \pi_{-eq}] = [\pi_{eq} \oplus 0_{-eq}]$
(ii) $[(\pi_{eq}, \theta_{eq}) \oplus (\pi_{-eq}, \theta_{-eq})] = [(\pi_{eq}, \theta_{eq}) \oplus (0_{-eq}, 0_{-eq})]$

Define a mapping $\psi : \eta_V^{-1}(\mathbb{V}^*) \longrightarrow \eta_N^{-1}(\mathbb{N}^*)$ by $\psi\Big([(\pi_{eq}, \theta_{eq}) \oplus (\pi_{-eq}, \theta_{-eq})]\Big) \longmapsto [\pi_{eq} \oplus 0_{-eq}]$. This mapping is well-defined by the following fact, which can be shown by applying Fact 2[58]:

Fact 3: Suppose $[(\pi_{eq}, \theta_{eq}) \oplus (\pi_{-eq}, \theta_{-eq})] \in \eta_V^{-1}(\mathbb{V}^*)$. Then $[\pi_{eq} \oplus 0_{-eq}] \in \eta_N^{-1}(\mathbb{N}^*)$.

We claim that ψ is an injection. Indeed, suppose $\psi\Big([(\pi_{eq}, \theta_{eq}) \oplus (\pi_{-eq}, \theta_{-eq})]\Big) = \psi\Big([(\pi'_{eq}, \theta'_{eq}) \oplus (\pi'_{-eq}, \theta'_{-eq})]\Big)$, i.e., $[\pi_{eq} \oplus 0_{-eq}] = [\pi'_{eq} \oplus 0_{-eq}]$, whence by Fact 1, $\pi_{eq} = \pi'_{eq}$. Then

$$\begin{aligned} [(\pi_{eq}, \theta_{eq}) \oplus (\pi_{-eq}, \theta_{-eq})] &= [(\pi_{eq}, \theta_{eq}) \oplus (0_{-eq}, 0_{-eq})] \\ &= [(\pi'_{eq}, 0_{eq}) \oplus (0_{-eq}, 0_{-eq})] \\ &= [(\pi'_{eq}, \theta'_{eq}) \oplus (0_{-eq}, 0_{-eq})] \\ &= [(\pi'_{eq}, \theta'_{eq}) \oplus (\pi'_{-eq}, \theta'_{-eq})], \end{aligned}$$

hence ψ is injective.[59] Let $\varphi := \eta_N \circ \psi \circ \eta_V^{-1}$, whence $\varphi : \mathbb{V}^* \longrightarrow \mathbb{N}^*$ is an injection, as claimed.

(ii) Consider $\psi : \eta_V^{-1}(\mathbb{V}^*) \longrightarrow \eta_N^{-1}(\mathbb{N}^*)$ in the proof of part (i). First, it is clear that $\varphi := \eta_N \circ \psi \circ \eta_V^{-1}$

[57] Namely, from Fact 1, it suffices to show that the left- and right-hand sides induce the same $\mathcal{P}$. But note that assigning 0_{-eq} (or $(0_{-eq}, 0_{-eq})$) ensures $V(\omega) = 0$, $(\omega \in \Omega_{-eq})$ hence the incentive compatibility conditions ensure that the off-equilibrium states continue to be off-equilibrium states. Furthermore, since these off-equilibrium states are never reached, other state values ($V(\omega)$, $\quad \omega \in \Omega_{eq}$) remain unaffected, hence the corresponding incentive compatibility conditions are preserved on π_{eq} (or (π_{eq}, θ_{eq})), and therefore, $\mathcal{P}$ is preserved.

[58] Namely, $[(\pi_{eq}, \theta_{eq}) \oplus (\pi_{-eq}, \theta_{-eq})] = [(\pi_{eq}, 0_{eq}) \oplus (0_{-eq}, 0_{-eq})] = [(\pi_{eq}, 0_{eq}) \oplus (0_{-eq}, 0_{-eq})]$ where the first equality follows from the definition of VPNE and the second equality is from Fact 2-(ii). Fact 3 then follows by comparing (3)-(4) with (9)-(11).

[59] Here, the second and third equalities are from the definition of VPNE, while the first and fourth equalities apply Fact 2-(ii).

preserves *trading* strategies since

$$\begin{aligned}\psi \circ \eta_V^{-1}(v) &= \psi\big([(\pi_{eq}, \theta_{eq}) \oplus (\pi_{-eq}, \theta_{-eq})]\big)\\ &= [\pi_{eq} \oplus 0_{-eq}]\\ &= [\pi_{eq} \oplus \pi_{-eq}],\end{aligned}$$

where the last equality applies Fact 2-(ii). Secondly, to show that the probability transitions are also preserved, first note again that:

$$\begin{aligned}\eta^{-1}(v) &= [(\pi_{eq}, \theta_{eq}) \oplus (\pi_{-eq}, \theta_{-eq})]\\ &= [(\pi_{eq}, \theta_{eq}) \oplus (0_{-eq}, 0_{-eq})]\\ &= [(\pi_{eq}, 0_{eq}) \oplus (0_{-eq}, 0_{-eq})].\end{aligned}$$

As equilibrium strategies, $[(\pi_{eq}, 0_{eq}) \oplus (0_{-eq}, 0_{-eq})]$ must satisfy (9)-(11) and, by definition, also generates $P^{\pi,\theta}$. By construction, (9)-(11) devolves into (3)-(4) given $\theta \equiv 0$. Hence $[\pi_{eq} \oplus \pi_{-eq}] = [\pi_{eq} \oplus 0_{-eq}] = \psi \circ \eta^{-1}(v))$ must also satisfy (3)-(4) and, by definition, also generates P^{π}. Meanwhile, note that

$$P^{\pi,\theta}\Big|_{\theta\equiv 0} = P^{\pi}.$$

Hence, φ preserves transition probabilities. And since we have previously shown that φ also preserves trading strategies, we conclude that φ preserves all allocations in equilibrium. □

Proof of Lemma 3

Proof. First, note that in any NE, a transition of state from ω_1 (most skewed) to ω_N (most equal) is never feasible in any consecutive time period, yet this transition ($1 \to N$) occurs whenever the quorum is reached in the "joint deviation" model. Therefore, it suffices to show an example whereby the quorum is inevitably attained.

Consider a NE where state 1 is 'recurrent'.[60] Note that $V(\omega) = \beta(1-k)V(\omega_N)$ whenever Q is attained on ω, and also, $V(\omega_1) < V(\omega_N)$ holds.[61] But for any set of parameters (β, k) such that $V(\omega_1) < \beta(1-k)V(\omega_N)$ holds, the quorum is inevitably attained, because any agent in state 1 will choose $\theta = 1$, and there are $N-1(\geq Q)$ such agents. □

[60] That is, state 1 arises with non-zero probability in equilibrium. We know such a NE exists, for example from our result in 3.2.1.

[61] An easy proof follows by contradiction.

Proof of Lemma 4

Proof. This follows directly from the incentive compatibility conditions (10) and definition of $\mathcal{Q}_{VI}$.

□