Bridging International Comovements in Inflation and Interest Rates

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Abstract

International comovements in inflation and interest rates have been separately documented by numerous empirical studies. In this paper we bridge the gap by investigating common natures that simultaneously generate these international comovements. To this end, we estimate a high-dimensional dynamic common factor model of national inflation rates, short-, and long-term bond yields. From the results based on 30 OECD countries' data, we find two macroeconomic factors that link the two global macroeconomic comovements for the past two decades. One is the global real activity, and the other is the global inflation fluctuations that are orthogonal to the global commodity price. (JEL classification: E31, E43, F62)

Keywords: Dynamic common factor model, variance decomposition, business cycles, commodity prices

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1 Introduction

Nominal interest rates and CPI inflation rates in different currencies have co-moved. These international comovements been separately documented by numerous empirical studies. Regarding the inflation rates, for example, Ciccarelli and Mojon (2010) report that one global inflation factor can explain nearly 70% of 22 OECD countries' inflation variation. Parker (2017) also finds that the synchronization of inflations is evident among advanced countries, while country-specific factors are more important in the middle or low-income countries' inflation rates.[1].

On the other hand, Altansukh, Becker, Bratsiotis, and Osborn (2017) analyze sectoral inflation rates and find that core and energy inflation rates drive the long-run and short-run cross-country comovements, respectively.[2]. Mikolajun and Lodge (2016) argue that the global inflation factor is a proxy of inflation expectation, which is the reason why it can substantially explain individual national inflation rates. Finally, Ha, Kose, and Ohnsorge (2019) find that a global factor's ability to explain the inflations of advanced countries as well as developing countries has increased over time.

As one of the first studies about the international interest rate comovement, Sutton (2000) investigates the comovement of 10-year sovereign bond yields of five major countries, and find that the cross-country excessive bond returns are highly correlated[3]. Afterward, Diebold, Li, and Yue (2008) estimate a dynamic common factor model of cross-country yield curves in a dynamic Nelson-Siegel framework, and quantify the degree of global yield curve comovement. In their model, each country's yield curve is decomposed into a global factor and country-specific factor. They provide strong evidence that a global yield curve factor exists and its effect differs across countries. Del Negro, Giannone, Giannoni, and Tambalotti (2018) also find that there exists a common trend in G7 real interest rates and it has been more apparent since late 1970. Recently, Takáts and Vela (2014) report that since the 2008 global financial crisis long-term bond yields

[1]He shows that the low development of the financial markets or low transparency of the central bank can increase the influence of the country-specific factors.

[2]Monacelli and Sala (2009) also show that a global inflation factor is found in disaggregated CPI data across country.

[3]The major countries are the United States, Japan, Germany, United Kingdom, and Canada.

have been more synchronized between the U.S. market and emerging markets.

Some studies argue that the two comovement phenomena may be related. For example, Chin, Graeve, Filippeli, and Theodoridis (2018) show that monetary policy accommodating foreign inflation can make national long-term yields co-move. Similarly, Jotikasthira, Le, and Lundblad (2015) find that global inflation is a relevant state factor to explain the international comovement in interest rates. Meanwhile, Dahlquist and Hasseltoft (2013) document that the global risk premium is related to the global business cycles. It means that interest rate comovement and inflation comovement can be related through global real activity.

Table 1: **Correlations between Principal Components** The table presents the correlations among the first principal components of the national short-term bond yields, long-term bond yields, and CPI inflation rates, which are denoted by $PC(S)$, $PC(L)$, and $PC(\pi)$, respectively. The results in column (b) are based on the detrended data.

	(a) before detrending			(b) after detrending		
	$PC(L)$	$PC(S)$	$PC(\pi)$	$PC(L)$	$PC(S)$	$PC(\pi)$
$PC(L)$	1			1		
$PC(S)$	0.8621	1		0.4990	1	
$PC(\pi)$	0.6053	0.7113	1	0.4622	0.6262	1

Our study is motivated by the fact that despite strong empirical evidence of the two international comovements, few studies have analyzed them in one framework. Theoretically, bond yields with different maturities and inflation rates are intimately related in a general equilibrium framework. Especially, short- and long-term interest rates are closely related through the market expectation hypothesis, and the inflation rate is a primary determinant of the policy rate for each country. A principal component analysis (PCA) also suggests the possibility that there exists a common driving factor to generate the cross-country comovements simultaneously. We compute the first principal component (PC) of 30 OECD countries' quarterly inflation rates ($PC(\pi)$), the first PC of the short-term rates ($PC(S)$), and the first PC of the long-term bond yields ($PC(L)$), which can be regarded as empirical proxies of global inflation, short-term interest rate, and long-term interest rate factors, respectively.[4] The correlation between ($PC(\pi)$)

[4]The long-term interest rates are ten-year government bond yields, and the short-term interest rates

and ($PC(S)$) is 0.71 and the correlation between ($PC(\pi)$) and ($PC(L)$) is 0.61, which strongly suggests the presence of a common driving nature.[5] On the other hand, the dynamics of the first PCs seem to have downward trends, which may cause a spurious correlation. However, Table 1 shows that the presence of the common factors and their high correlations among the first PCs are robust to the detrended data.

In this paper, we attempt to bridge the gap between the two global comovements by identifying and quantifying the role of common driving factors. To this end, we estimate a high-dimensional dynamic common factor model of national inflation rates, short-term interest rates, and long-term bond yields. As illustrated by Table 2 and Figure 1, seven common factors are identified in the model: a common factor of long-term yields, short-term rates, and inflation rates (global factor), a common factor of long-term yields and short-term rates (interest rate factor), a common factor of long-term yields and inflation rates (LR-inflation factor), a common factor of short-term rates and inflation rates (SR-inflation factor), a common factor of long-term yields (long-rate factor), a common factor of short-term rates (short-rate factor), a common factor of inflation rates (inflation factor). All these factors are assumed to be mutually independent. Our focus is centered on the role of the three common factors among interest rates and inflation rates: the global factor, LR-inflation factor, and SR-inflation factor. Their effect on the cross-country macroeconomic variables is quantified by the variance-decomposition.

Our empirical findings based on the 30 OECD countries data over the period from the second quarter of 2002 to the fourth quarter of 2019 can be summarized as follows. First, we find that the cross-country comovement in each macroeconomic variable is evident. The seven common factors estimated from our model explain variations of cross-country long-term yields, short-term rates, and inflations by 86.9%, 80.4% and 74.1%, respectively, on average. Second, more importantly, the common factors explain the majority of the national inflation and interest rates on average. The global

are three-month money market rates. The inflations are calculated by the year on year growth rates of the consumer price indices. The sample period is from the second quarter of 2002 to the fourth quarter of 2019 in which all data are available. Those are available from the OECD Main Economic Indicators database

[5] All data are normalized by z-score to prevent a situation in which a particular dependent variable is weighted heavily in the common factor estimation simply because of its large variance.

Table 2: **Description of the common factors** This table represents specific meanings of common factors.

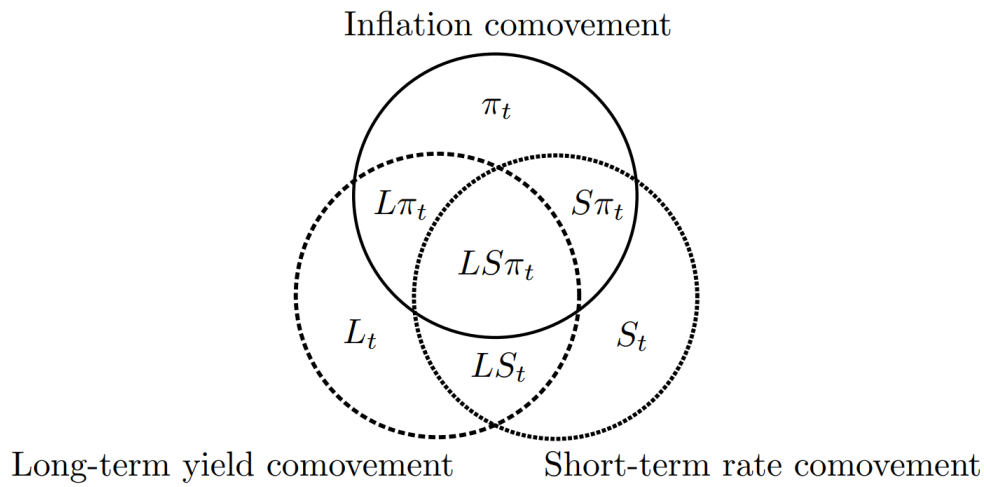
Notation	Factor	Description
		A common factor of
$LS\pi_t$	global factor	long-term yields, short-term rates, and inflations
LS_t	interest rate factor	long-term yields and short-term rates
$L\pi_t$	LR-inflation factor	long-term yields and inflations
$S\pi_t$	SR-inflation factor	short-term rates and inflations
L_t	long-rate factor	long-term yields
S_t	short-rate factor	short-term rates
π_t	inflation factor	inflations

and SR-inflation factors, which are common factors among the national inflation rates and short-term rates, explain 38.8% of the national inflation rates and 34.5 % of the national short-term rates on average. Further, the global factor and LR-inflation factor account for 38.8% of the national inflation rates 48.5% of the national long-term rates on average. In other words, it turns out that the international interest rate comovement can be decomposed into the components related and unrelated with the global inflation synchronization.

Third, we uncover two bridges to connect the two synchronization phenomenons. The first bridge is the global real growth rate which turns out to be closely related with the global factor. The second bridge is the non-commodity price global inflation, which is the fluctuations of the inflation rates not explained by the global commodity price index. Using various comparisons, we show that the non-commodity price global inflation is highly correlated with the SR-inflation and LR-inflation factors. This seems to be attributed to the fact that the central banks react more aggressively to aggregate demand shocks rather than cost-push shocks.

The rest of this paper is organized as follows. The next section illustrates the data used for our empirical work. Before we get into high-dimensional common factor analysis, we study the two comovement phenomena' natures using simple P.C. analysis in Section 3. Section 4 presents the high dimensional dynamic common factor model that we estimate. The estimation results and their interpretation are provided in Section 5. Section 6 concludes the paper.

Figure 1: **Description of the Common Factors** The solid circle represents the co-movement of the cross-country inflation rates. The dashed and dotted circles indicate the comovement of the cross-country long-term and short-term interest rates, respectively. $LS\pi_t$ is the global factor that jointly influences on all interest rates and inflation rates across countries at time t. $L\pi_t$, $S\pi_t$, and SL_t are the LR-inflation, SR-inflation factor, and LR factors, respectively. Finally, π_t, S_t , and L_t are the inflation, short-rate, and long-rate factors, respectively. All these common factors are assumed to be mutually independent.



2 Model

2.1 State-Space Representation

In this section, we represent our dynamic common factor model of the national short-term interest rats, long-term interest rates, and inflation rates in a state-space form. For this, we denote the N-dimensional vectors of long-term yields, short-term interest rates, and inflation rates at time t by L_t, S_t, and π_t, respectively, where $N(=30)$ is the number of the countries considered here.

As mentioned earlier, we assume the seven common factors, denoted by F_t, to generate the dynamics of the national macroeconomic variables (i.e, L_t, S_t, and π_t) is as

follows:

$$\underbrace{\begin{pmatrix} L_t \\ S_t \\ \pi_t \end{pmatrix}}_{y_t} = \underbrace{\begin{bmatrix} \alpha^G & \alpha^Y & \alpha^L & O_{N\times1} & \alpha^* & O_{N\times1} & O_{N\times1} \\ \psi^G & \psi^Y & O_{N\times1} & \psi^S & O_{N\times1} & \psi^* & O_{N\times1} \\ \delta^G & O_{N\times1} & \delta^L & \delta^S & O_{N\times1} & O_{N\times1} & \delta^* \end{bmatrix}}_{V} \times \underbrace{\begin{pmatrix} LS\pi_t \\ LS_t \\ L\pi_t \\ S\pi_t \\ L_t \\ S_t \\ \pi_t \end{pmatrix}}_{F_t} + \underbrace{\begin{pmatrix} \epsilon_t^L \\ \epsilon_t^S \\ \epsilon_t^\pi \end{pmatrix}}_{\epsilon_t}.$$

To identify the common factors as in Table 2, we impose zero restrictions on the factor loading, V. For example, the LR-inflation factor, $L\pi_t$ is allowed to influence on all long-term yields and inflation rates, but not on the short-term interest rates. Thus, $L\pi_t$ is identified as a common driving factor of the long-term interest rate and inflation comovements excluding the global factor effect. Meanwhile, the factor loading of the long-rate factor, L_t is non-zero for the national long-term interest rates only, so it captures the long-term interest rate comovement excluding the effects of $L\pi_t$ and $LS\pi_t$. The common factors are assumed to generated from mutually independent and stationary first-order autoregressive (AR(1)) processes,

$$F_t | F_{t-1}, \Phi, \Omega_F \backsim \mathcal{N}\left(\Phi F_{t-1}, \Omega_F\right),$$

where Φ and Ω_F are diagonal matrices.

In addition, to identify the common factor shock variances we fix one element of each column of the factor loading matrix V at 0.25. However, the element should be carefully chosen. If the element corresponding to the country variable that is not affected by the common factor, the common factor is poorly identified and it is not precisely estimated. This is a typical problem that arises in a high-dimensional dynamic common factor model, and we deal with this issue as the following. Given that the first column is the coefficients of the $LS\pi_t$, we compute the 1st PC of y_t. Then, we compute the correlations between the PC and each variable of y_t. The element corresponding the variable with the largest correlation is chosen to be fixed at 0.25. For the second column, we compute the 1st PC of (L_t, S_t) and estimate the correlations between the PC and each variable of (L_t,

S_t). As in the first column, we fix the element of the variable with the largest correlation. For the last column, we obtain the 1st PC of π_t and compute the correlation between the PC and each variable of π_t. The element of the variable with largest correlation is fixed. This is repeated for the rest of the columns.

The vectors, ϵ_t^L, ϵ_t^S, and ϵ_t^π are the N-dimensional country-specific factors at time t for the long-term interest rates, short-term interest rates, and inflation rates, respectively. Given $3N \times 3N$ diagonal matrices, Θ and Ω_ϵ, they are also assumed to follow mutually independent and stationary AR(1) processes,

$$\epsilon_t | \epsilon_{t-1}, \Theta, \Omega_\epsilon \backsim \mathcal{N}\left(\Theta \epsilon_{t-1}, \Omega_\epsilon\right).$$

2.2 Prior Distribution

Our econometric approach is Bayesian, and we complete our modeling by specifying the prior distributions for the model parameters. Table 2 summarizes the prior distributions. Basically, Our prior distributions are conjugate and spread out, so that the posterior distribution is mostly updated by the information in the data rather than in the prior. First of all, regarding the AR(1) coefficients, our prior belief is that the all factors including F_t and ϵ_t are positively autocorrelated, not explosive. Accordingly, all diagonal elements off Φ and Θ are assumed to be generated from a truncated normal $\mathcal{N}_{[-1,1]}(0.5, 0.3^2)$. Next, the common factor shock variances, which are the diagonal elements of Ω_F, are assumed to follow an inverse-gamma distribution, $\mathcal{IG}(14, 12)$. The prior distributions for the idiosyncratic factor shock variances in Ω_ϵ are given by $\mathcal{IG}(14, 3/4)$.[6].

Finally, we believe that the factor loadings are a bit more likely to have positive values than negative values, and assume a normal distribution, $\mathcal{N}(0.5, 0.25)$ for the factor loading parameters. The positive factor loadings imply positive correlations among long-term yields, short-term rates, and inflation rates. Such prior belief is based on central

[6]As mentioned at section 2.1, some factor loadings are fixed to 0.25. On the contrary, the factor loading of idiosyncratic factor ϵ_t is fixed at one. Therefore, if we believe that ϵ_t and common factors have the same explanatory power on a dependent variable,
which is defined as (factor loading)$^2 \cdot$ (factor's unconditional variance),
then the error variance of the common factor must be 16 times larger than error variance of ϵ_t. Since the prior distribution of error variance ϵ_t is $\mathcal{IG}(14, 3/4)$, we impose the prior distribution of common factor's error variance as $\mathcal{IG}(14, 0.75 \times 16)$.

banks' macroeconomic stabilization policy and expectation hypothesis of a standard arbitrage-free affine bond pricing theory. Central banks react to output gap and inflation fluctuations due to aggregate demand shocks, which results in a comovement between the inflation rates and short-term interest rates in the same direction. On the other hand, the expectation hypothesis indicates a positive correlation between long-term yields and short rates, because a long-term yield can be expressed as a linear combination of the current and expected path of short-term rates.

Figure 2: **Prior distribution** This table presents the prior distributions of the factor loadings, AR(1) coefficients, and shock variances. $\sigma^2_{F,i}$ is the common factor shock variance, where $\Omega_F = diag(\sigma^2_{F,1}, \sigma^2_{F,2}, .., \sigma^2_{F,7})$. $\sigma^2_{\epsilon,i}$ is an idiosyncratic shock variance, where $\Omega_\epsilon = diag(\sigma^2_{\epsilon,1}, \sigma^2_{\epsilon,2}, .., \sigma^2_{\epsilon,90})$. $\mathcal{IG}$ indicates an inverse-gamma distribution.

Parameters	Prior distributions	Parameters	Prior distributions
Factor loadings	$\mathcal{N}$(0.5,0.25)	σ^2_F	$\mathcal{IG}$(14,12)
AR(1) coefficient	$\mathcal{N}_{[-1,1]}$(0.5,0.09)	σ^2_ϵ	$\mathcal{IG}$(14,3/4)

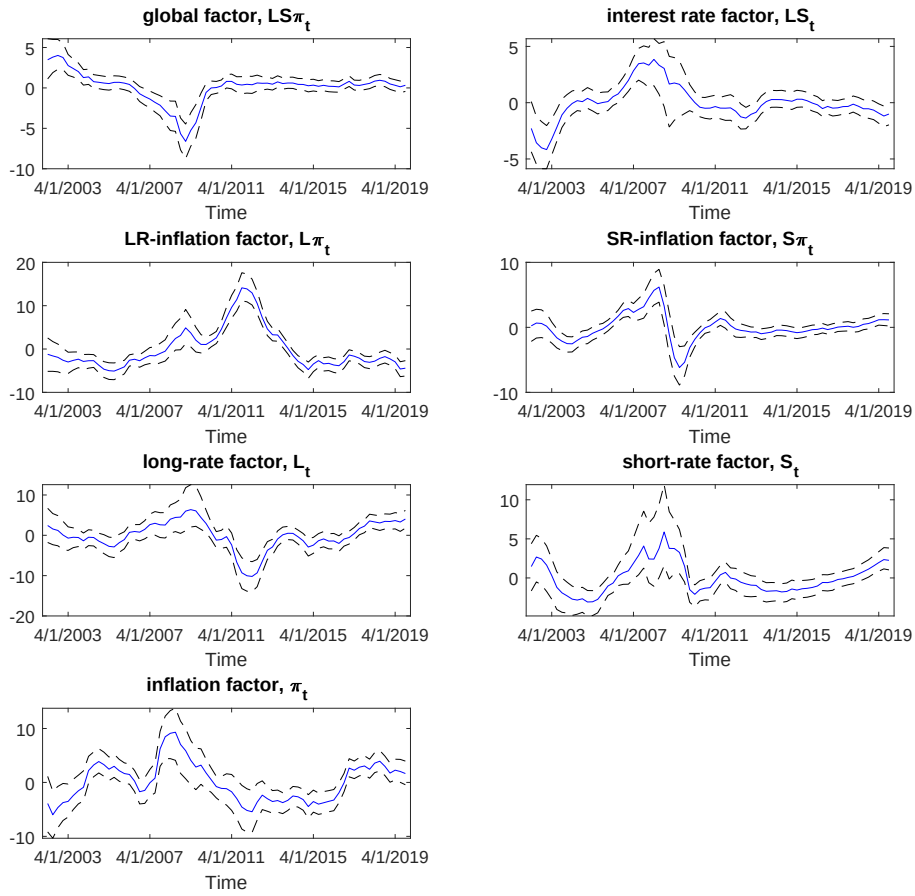
3 Estimation Results

We simulate the posterior distributions of the parameters and common factors via a Gibbs sampling. We run 2×10^6 Markov chain Monte Carlo iterations, and discard the first $10,000$ posterior samples. Further, we retain every 100th draws and use them for the posterior inference. The largest inefficiency factor is at most 3.8, strongly indicating the convergence of the Markov chain. The detailed discussion of the posterior simulation is provided in Appendix A.

3.1 The Common Factors

Figure 3 plots the posterior mean and 95% credibility intervals of the seven common factors over time. The common factors seem to be precisely estimated. Our key interests are centered on the three common factors, namely, $LS\pi_t$, $L\pi_t$, and $S\pi_t$ that drive the international inflation-interest rate comovements. We investigate the existence of the common factors, and then quantify their roles in explaining the dynamics of the national

Figure 3: **The common factors** The black dashed lines are 95% credibility intervals. The blue solid lines are the posterior medians.



inflation and interest rates.

Figure 4: **The factor loadings of the global factor** ($LS\pi_t$) The outer intervals with broken lines and the blue boxes are 90% and 60% posterior credibility intervals, respectively. The middle red lines are the posterior median.

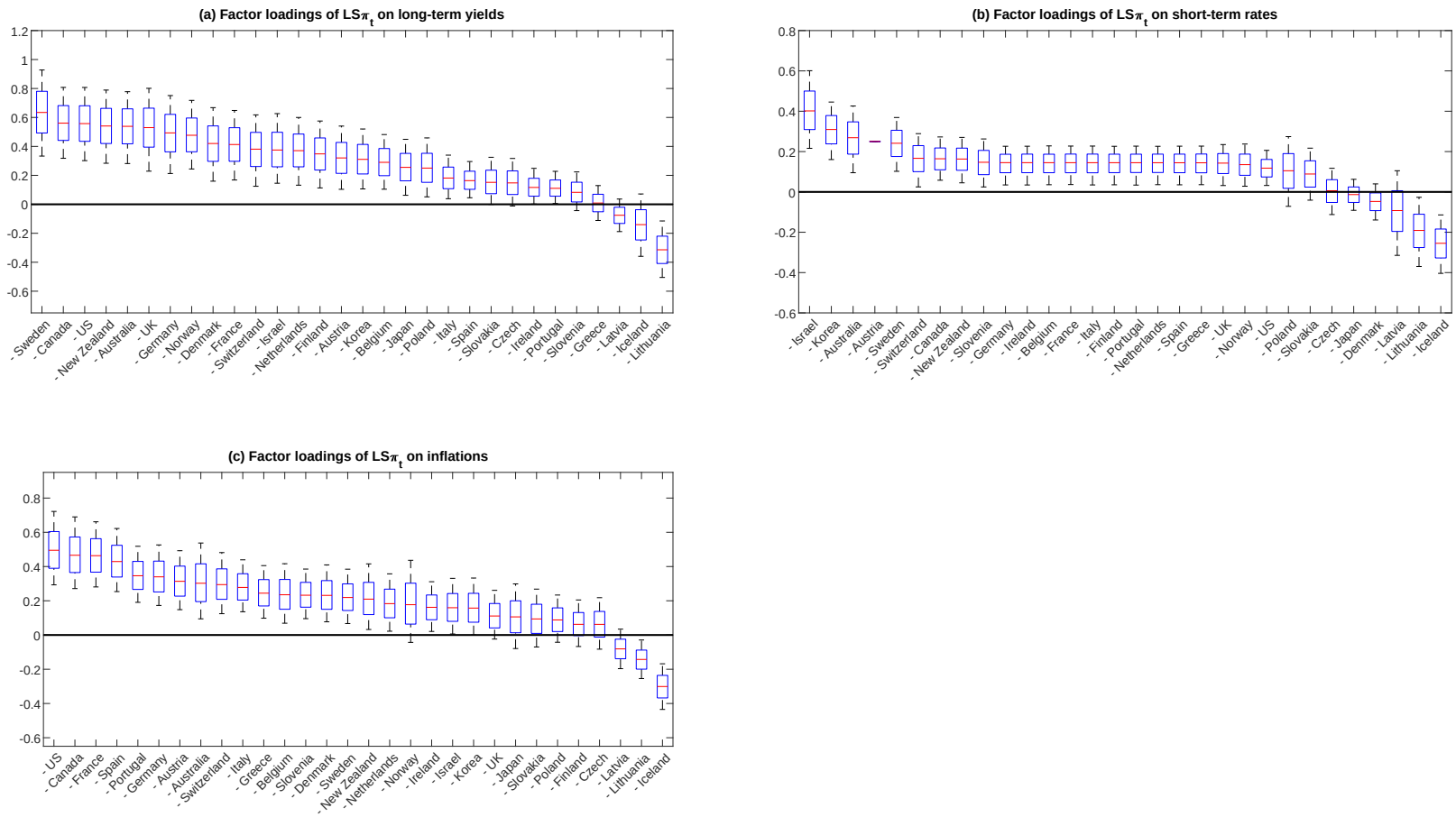
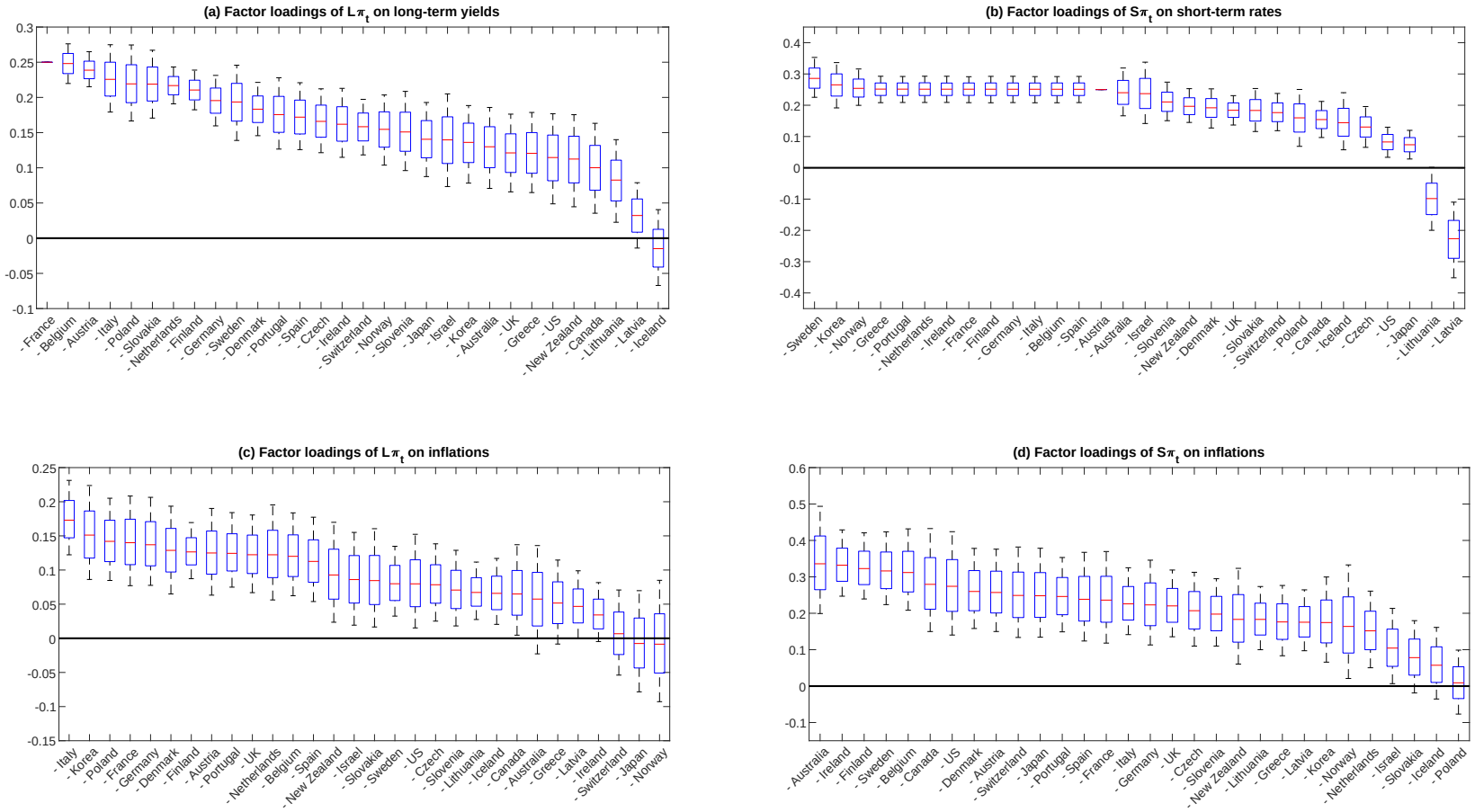


Figure 5: **The factor loadings of the LR-inflation factor ($L\pi_t$) and SR-inflation factor ($S\pi_t$)** The outer intervals with broken lines and the blue boxes are 90% and 60% posterior credibility intervals, respectively. The middle red lines are the posterior median.



The existence of the common factors can be tested by the posterior credibility intervals of the factor loadings across countries. If the majority of the credibility intervals does not contain zero, it can be said that the corresponding common factor is well-identified and exists. Figures 4 and 5 report the results of the factor loading credibility intervals for $LS\pi_t$ and $(L\pi_t,\ S\pi_t)$, respectively. Outer intervals with broken lines in each sub-figure show 90% credibility intervals. These figures show that most credibility intervals are strictly positive for all three common factors. Therefore, it is evident that there exist common factors driving the international comovements in inflation and interest rates.

Next, we turn to measuring the roles of the common factors by the variance decomposition. Figures 6 and 7 present the portions of the global factor, LR-inflation factor and SR-inflation factor for each country. Table 3 reports the average posterior means of the portions over the countries. The most noticeable finding is that the variations of the long-term yields, short-term rates, and inflation rates are explained by the three common factors to a large extent. The portions of the long-term yields, short-term rates, and inflation rates driven by the three factors are 68.35%, 62.85%, and 56.51%, respectively, on average.

Especially, the LR-inflation factor accounts for 28.27% of the long-term yields, the interest rate factor explains 29.94% of the short-term rates, and the global factor explains 20.46% of the inflation rates on average. Although the most relevant common factors differ across the macro variables, Table 3 shows that none of the factors seem to dominate the others. Consequently, the three common factors generate substantial comovements of the interest rates and inflation rates in the same direction. It follows that the international interest rate comovement can be the result of international inflation comovement, or vice versa. The casuality can be analyzed by a structural approach, given that our approach is purely statistical based on a reduced form model. In addition, it is noteworthy that as shown by Figures 6 and 7, the portions explained by each common factor are quite heterogeneous across the countries for each macro variable, although the source of the heterogeneity is not discussed here.

Figure 6: **Variance decomposition: the global factor** This figure plots the portions of the national interest rates and inflation rates explained by the global factor. The outer intervals with broken lines and the blue boxes are 90% and 60% posterior credibility intervals, respectively. The middle red lines are the posterior median.

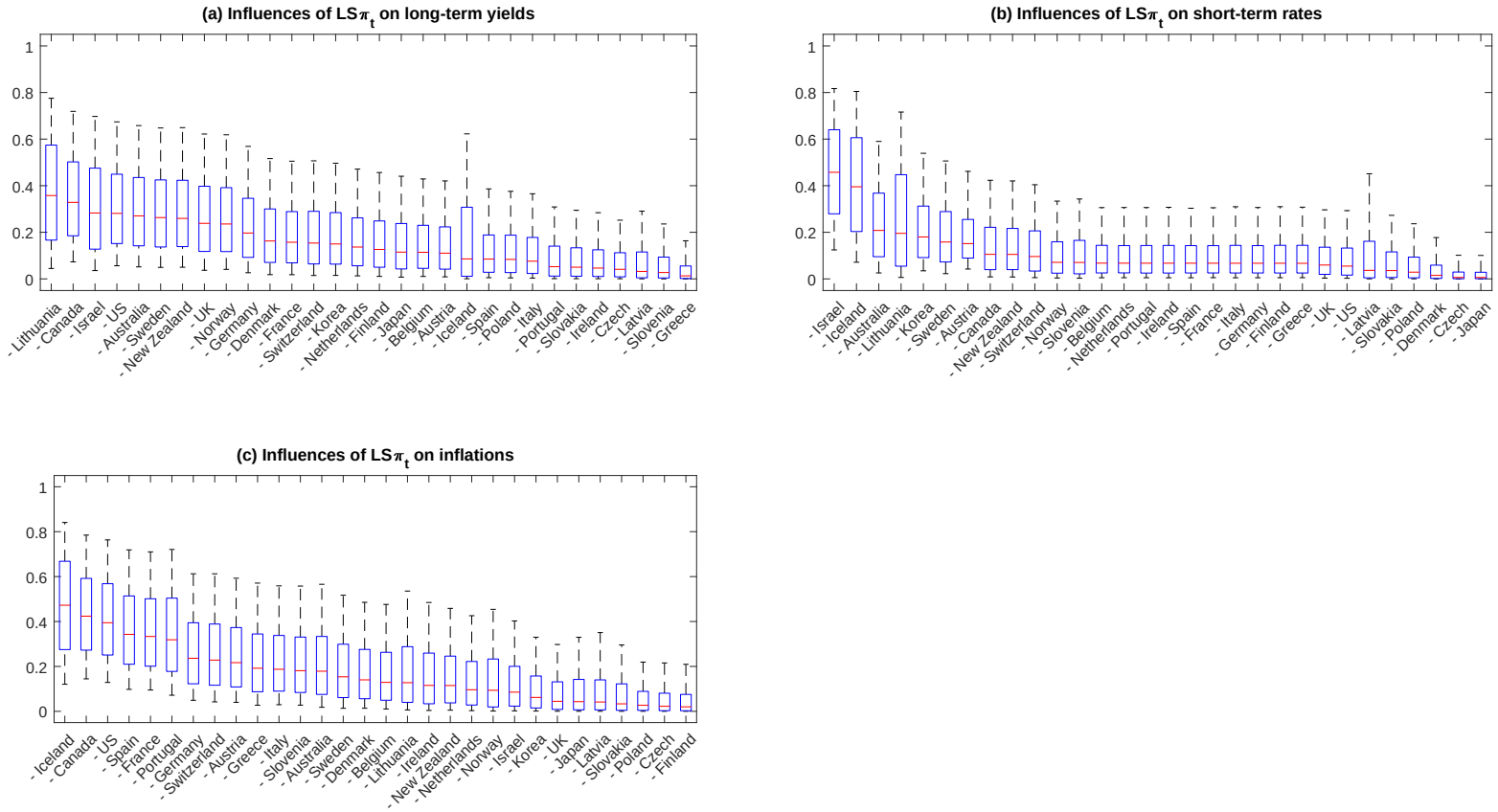


Figure 7: **Variance decomposition: the LR-inflation and SR-inflation factors** This figure plots the portions of the national interest rates and inflation rates explained by the LR-inflation and SR-inflation factors. The outer intervals with broken lines and the blue boxes are 90% and 60% posterior credibility intervals, respectively. The middle red lines are the posterior median.

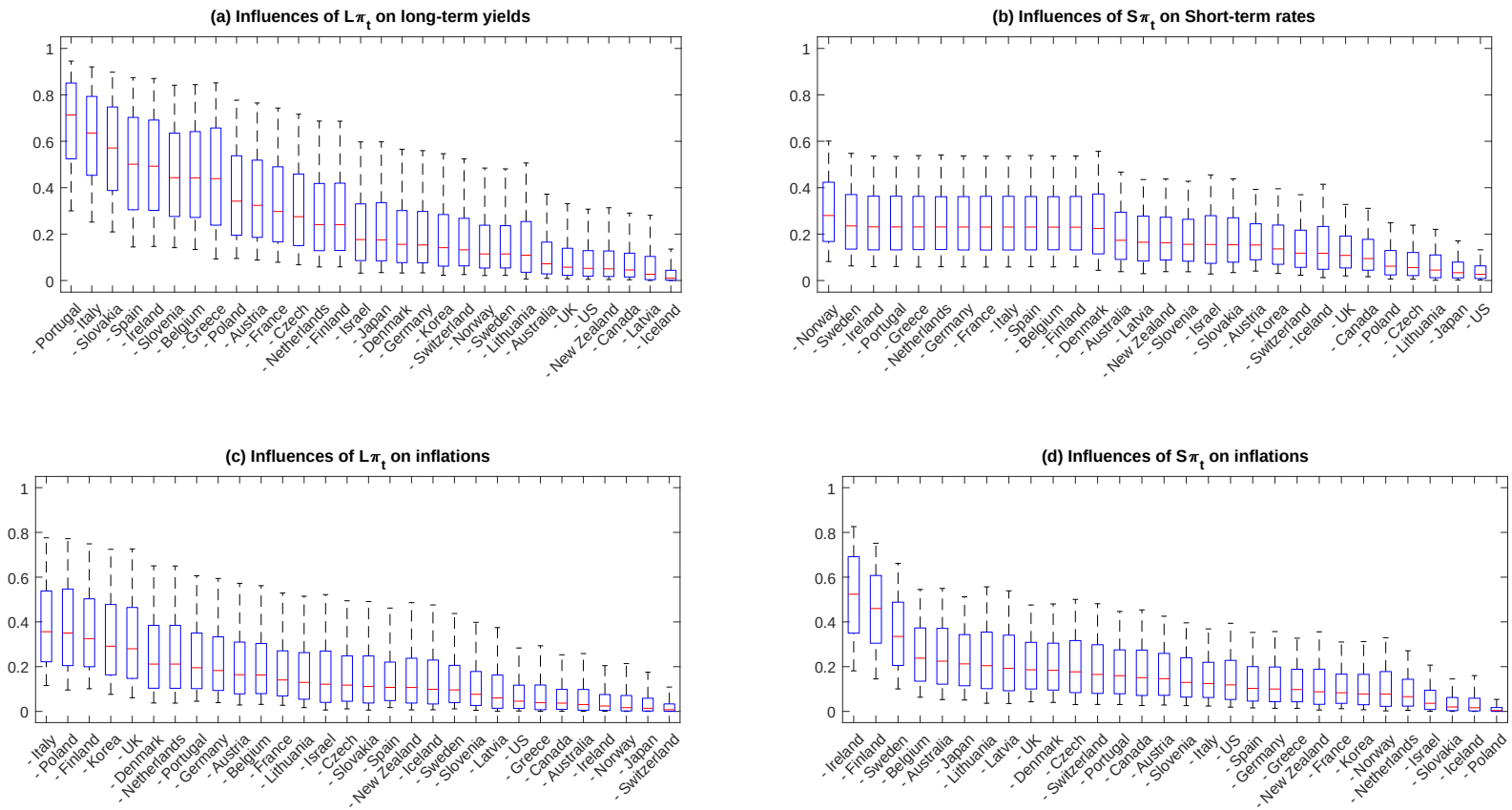


Table 3: **Average variance decomposition** This table reports the variance decomposition averaged over the countries using the posterior means of the portions across countries. The values in parentheses are the standard deviations.

Dependent Variables \ Explanatory Variables	global factor ($LS\pi_t$)	interest rate factor (LS_t)	LR-inflation factor ($L\pi_t$)	SR-inflation factor ($S\pi_t$)	long-rate factor (L_t)	short-rate factor (S_t)	inflation factor (π_t)	idiosyncratic factor (ϵ_t)
long-term yield	18.68%	21.40%	28.27%		17.96%			13.69%
	(9.36%)	(11.17%)	(17.79%)		(9.06%)			(15.32%)
short-term rate	13.76%	29.94%		19.16%		17.21%		19.94%
	(9.87%)	(12.02%)		(7.28%)		(9.54%)		(19.64%)
inflation	20.46%		17.77%	18.28%			17.77%	25.72%
	(11.85%)		(10.32%)	(11.13%)			(9.46%)	(15.52%)

3.2 Interpretation

In this section, we show that the global factor estimate is closely related to the global real activity, and that to a large extent the LR-inflation and SR-inflation factor estimates reflect the portion of the global inflation fluctuations that is independent of global commodity prices. Consequently, the global real activity and the portion of the global inflation unrelated to commodity prices serve as a bridge to connect the international interest rate comovement and international inflation comovement.

3.2.1 The Global Factor

Here we discuss how the global real activity is theoretically related to the short- and long-term interest rates and inflation rates. Then, we provide empirical evidence that the global factor estimate is proxies of the global business cycles.

According to the canonical New Keynesian open economy model of Galí and Monacelli (2005), an open economy can be described with two equations; the Dynamic IS equation(DIS) and the New Keynesian Phillips Curve(NKPC).

$$DIS : \widetilde{y}_t = E_t[\widetilde{y}_{t+1}] - \frac{1}{\sigma}(S_t - E_t[\pi^H_{t+1}] - r^*_t) \tag{1}$$

$$r^*_t = \rho - \zeta_1 a_t + \zeta_2 E_t[\Delta y^W_{t+1}] \tag{2}$$

$$NKPC : \pi^H_t = \beta E_t[\pi^H_{t+1}] + \kappa \widetilde{y}_t \tag{3}$$

$$\pi_t = (1-\alpha)\pi^H_t + \alpha \pi^W_t + \alpha \Delta e_t \tag{4}$$

$$\Delta e_t = (1-\alpha)\sigma(\Delta y_t - \Delta y_t^W) + (\pi_t - \pi_t^W) \tag{5}$$

where $\mathbb{E}_t[\cdot]$ is the conditional expectation given the information available at time t, $\widetilde{y}_t$ is the (domestic) output gap, Δy_t is the (domestic) growth rate, S_t is the 3-month(one-period) bond yield, r_t^* is the (domestic) natural rate of interest, a_t is the (domestic) log productivity, Δy_t^W is the world growth rate, π_t is the (domestic) CPI inflation rate, π_t^H is the price inflation of domestically produced good, π_t^W is the global inflation, and Δe_t the rate of change in the effective exchange rate at time t. According to equation (5)[7], Δe_t reflects a deviation from the global economy. Therefore, the movements of Δe_t in each country are idiosyncratic.

Under the assumption $\lim_{j\to\infty} \beta^j \mathbb{E}_t[\pi_{H,t+j}] = 0$ with $0 < \beta < 1$, the below forward solution of equation (3) shows the domestic inflation is determined by the current and expected future output gaps.

$$\pi_t^H = \kappa \sum_{j=0}^{\infty} \beta^j \mathbb{E}_t[\widetilde{y}_{t+j}] \tag{6}$$

Also, by equation (4) and (6), CPI inflation is determined by the below expression.

$$\pi_t = (1-\alpha)\kappa \sum_{j=0}^{\infty} \beta^j \mathbb{E}_t[\widetilde{y}_{t+j}] + \alpha\pi_t^W + \alpha\Delta e_t \tag{7}$$

Suppose that given the inflation target π^*, the central bank controls the short-term policy rate according to the Taylor rule (Taylor (1993) and Clarida, Gali, and Gertler (2000)) such as

$$S_t = s^* + \rho_1(\mathbb{E}_t[\pi_{t+q}] - \pi^*) + \rho_2\widetilde{y}_t + u_t \tag{8}$$

The constant term, s^* is the target short-term rate when the expected inflation and output gap are equal to their target values, and u_t is the policy shock at time t. By combining the equation (8), (4), and (6), the Taylor equation can be re-written as

[7]This equation can be derived using equations in Galí and Monacelli (2005); The q_t equation under the equation (15) and the equation (29) in Galí and Monacelli (2005) are used.

follows.

$$S_t = s^* - \rho_1\pi^* + \rho_1\{\frac{\rho_2}{\rho_1}\widetilde{y}_t + (1-\alpha)\kappa\sum_{j=0}^{\infty}\beta^j\mathbb{E}_t[\widetilde{y}_{t+q+j}] + \alpha\mathbb{E}_t[\pi^W_{t+q} + \Delta e_{t+q}]\} + u_t \quad (9)$$

The national 10-year bond yield at time t can be decomposed into the market expectation hypothesis component and the term premium as the following:

$$L_t = \frac{1}{40}\sum_{j=0}^{39}\mathbb{E}_t[S_{t+j}] + TP_t, \quad (10)$$

where TP_t is term premium at time t. By combining the equations (10) and equation (9), the long-term interest rate can be expressed as

$$L_t = \frac{1}{40}\sum_{j=0}^{39}[s^* - \rho_1\pi^* + \rho_1\{\frac{\rho_2}{\rho_1}\mathbb{E}_t[\widetilde{y}_{t+j}] + (1-\alpha)\kappa\sum_{k=0}^{\infty}\beta^k\mathbb{E}_t[\widetilde{y}_{t+j+q+k}]\} \quad (11)$$

$$+ \alpha\rho_1\mathbb{E}_t[\pi^W_{t+j+q} + \Delta e_{t+j+q}] + \mathbb{E}_t[u_{t+j}]] + TP_t \quad (12)$$

The equation (9), (11), and (7) indicate that cross-country comovement of (current and expected future) output gaps can generate the inflation comovement and the interest comovement simultaneously. Furthermore, the expected future world growth can make the output gap comovement through the DIS equation (i.e., equation (1) and (2)). Therefore, the expected future world growth(i.e., global real activity) can be a common nature driving the joint dynamics of the short- and long-term interest rates and inflation rates worldwide.

Now we empirically demonstrate that our global factor estimate is intimately related to the current or expected real activities. First of all, Figure 8 plots several proxies of global real activity conditions along with the global factor.[8] The global factor estimate and the real GDP growth rates of the United States, G7, and OECD countries exhibit very similar time series dynamics. The correlations between the global factor and real growth rates are 0.72 (U.S.), 0.70 (G7), and 0.67 (OECD) during the sample period. This shows that the global factor contains the information about the current global real economic condition.

[8]The data for the real GDP are available from the OECD National Accounts Statistics database.

Figure 8: **Dynamics of real growth rates** Black broken line shows the estimated dynamics of the global factor $LS\pi_t$. Solid lines show year on year growth rates of real GDP.

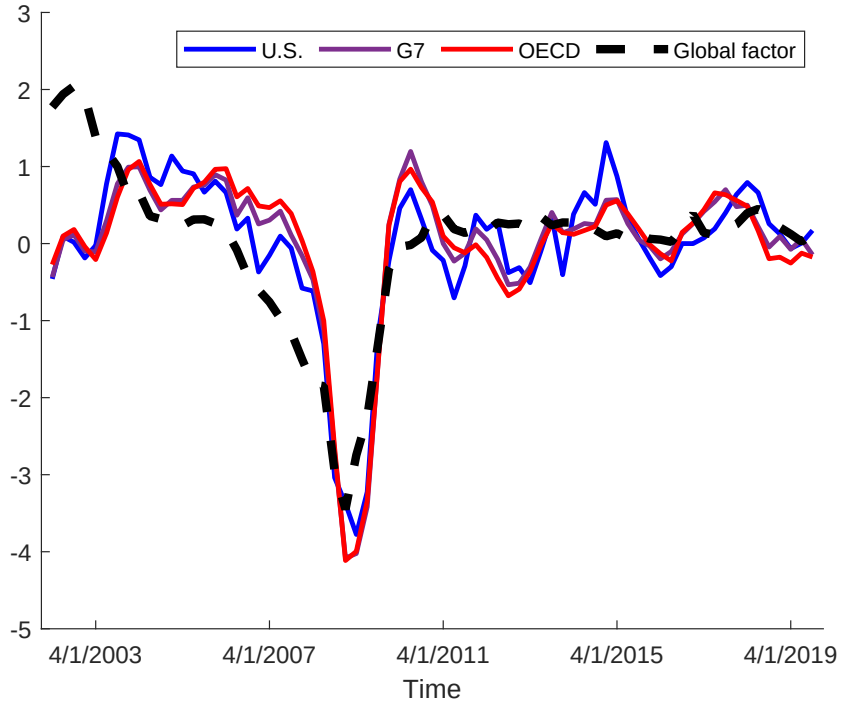


Table 4: **Estimation results of the predictive equation** The estimation results of the equation (13) are given. The numbers in parentheses represent 95% credible intervals. The equation is estimated using the Gibbs sampling with the Normal-Normal update and InverseGamma-InverseGamma update under uninformative prior. Bayesian R^2 is calculated according to Gelman et al. (2019).

horizon	cons	$LS\pi_t$	Bayesian R^2
3-month	1.8270	0.4978	0.2018
	(1.3661, 2.287)	(0.253, 0.7346)	(0.0587, 0.3592)
6-month	1.8390	0.4391	0.1900
	(1.4117, 2.2673)	(0.2188, 0.6604)	(0.05, 0.3488)
9-month	1.8487	0.4021	0.1909
	(1.4508, 2.2459)	(0.2, 0.6038)	(0.0498, 0.35)
1-year	1.8609	0.3717	0.1940
	(1.4915, 2.2291)	(0.1849, 0.5588)	(0.0513, 0.3550)
3-year	1.8398	0.2155	0.2049
	(1.6062, 2.0742)	(0.1036, 0.3269)	(0.0505, 0.3756)
5-year	1.7225	0.1280	0.1789
	(1.5458, 1.8992)	(0.0494, 0.2065)	(0.0274, 0.361)

Next, we examine whether the global factor reflects the expectation of the future global real activities. This can be easily done by testing the prediction ability of the global factor about the future real activities. For this, we estimate the following predictive model;

$$4\frac{100(\log y_{t+h} - \log y_t)}{h} = \eta_0 + \eta_1 LS\pi_t + u_{t+h}, \tag{13}$$

where y_t is the OECD real GDP and u_t is a forecasting error at time t. Table 4 presents the estimation results for the various forecasting horizons. The regression effects of the past global factors are statistically supported, and the R^2 is also large enough to say that the global factor contains information about future global real GDP. Consequently, we can conclude that the global factor is strongly associated with the current and future global real activity.

3.2.2 The LR-inflation Factor

For each country, the long-term bond yield can be expressed as the sum of the real long-term yield (l_t) and inflation compensation (π_t^e), as follows:

$$L_t = l_t + \pi_t^e.$$

The inflation compensation can also be decomposed into the inflation expectation and inflation risk premium,

$$\pi_t^e = \mathbb{E}_t[\pi_{t+1:t+40}] + \varphi_t,$$

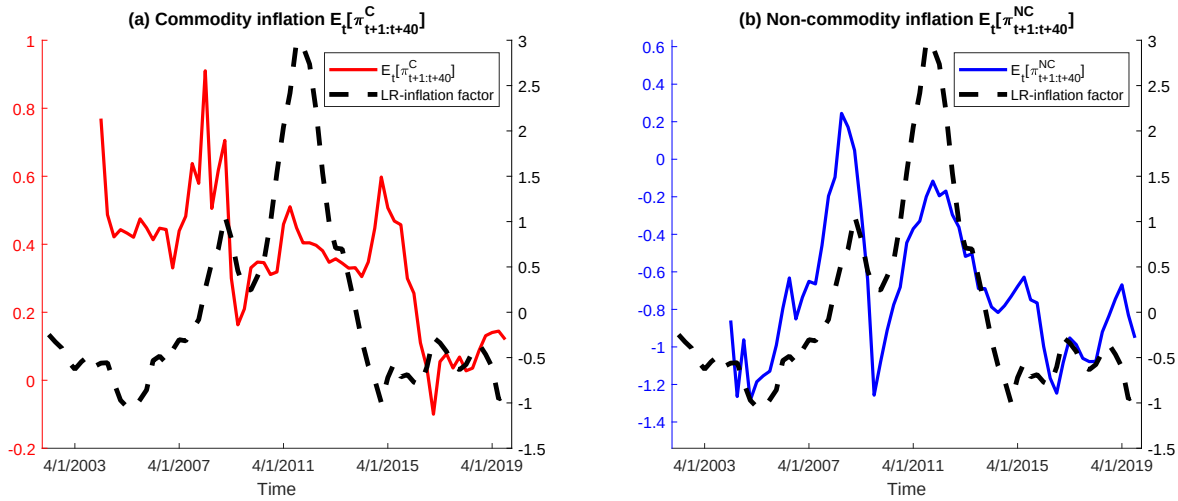
where

$$\mathbb{E}_t[\pi_{t+1:t+40}] \equiv \frac{1}{40}\mathbb{E}_t[\pi_{t+1} + \cdots + \pi_{t+40}]$$

is the average expected inflation over the next 40 quarters and φ_t is the inflation risk premium at time t. According to equation (4), CPI inflation is a function of domestic inflation, global inflation, and the exchange rate change rate. Therefore, inflation expectation also can be expressed with those three variables.

$$\mathbb{E}_t[\pi_{t+1:t+40}] = (1-\alpha)\mathbb{E}_t[\pi^H_{t+1:t+40}] + \alpha\mathbb{E}_t[\pi^W_{t+1:t+40}] + \alpha\mathbb{E}_t[\Delta e_{t+1:t+40}]$$

Figure 9: **Economic interpretations of the LR-inflation factor** The black dashed line shows the estimated dynamics of the LR-inflation factor. The solid red line and the solid blue line show $E_t[\pi^C_{t+1:t+40}]$ and $E_t[\pi^{NC}_{t+1:t+40}]$, respectively.



where, $\mathbb{E}_t[\pi^H_{t+1:t+40}] \equiv \frac{1}{40}\mathbb{E}_t[\pi^H_{t+1} + \cdots + \pi^H_{t+40}]$, $\mathbb{E}_t[\pi^W_{t+1:t+40}] \equiv \frac{1}{40}\mathbb{E}_t[\pi^W_{t+1} + \cdots + \pi^W_{t+40}]$, and $\mathbb{E}_t[\Delta e_{t+1:t+40}] \equiv \frac{1}{40}\mathbb{E}_t[\Delta e_{t+1} + \cdots + \Delta e_{t+40}]$.

Finally, we note that the global inflation can be decomposed into two orthogonal components. One component that is related to commodity prices and we denote this by $\pi^C_{i,t}$. The other component is the non-commodity global inflation, denoted by $\pi^{NC}_{i,t}$, which is independent of the commodity prices. As a result, the long-term bond yield is determined by a sum of the six components as the following:

$$L_t = l_t + (1-\alpha)\mathbb{E}_t[\pi^H_{t+1:t+40}] + \alpha\mathbb{E}_t[\pi^C_{t+1:t+40}] + \alpha\mathbb{E}_t[\pi^{NC}_{t+1:t+40}] + \alpha\mathbb{E}_t[\Delta e_{t+1:t+40}] + \varphi_t.$$

Here we empirically show that the LR-inflation factor have information about the expected non-commodity global inflation component. The first step is extracting the non-commodity global inflation component from the global inflation. Secondly, we use the first principal of the national inflation rates, $PC(\pi)$ as a proxy of the global inflation, and regress it on a constant and the growth rate of the global commodity price index (GCPI).[9] Accordingly, the fitted values and residuals can be treated as commodity price-driven global inflation and non-commodity global inflation components, respectively.

[9]The GCPI data is obtained from the FRB of St. Louis FRED.

The final step is to estimate the expected global inflation components over time. The expectations are calculated by estimating the univariate AR(4) forecasting regression. Figure 9 reports the estimation results of the expected inflation components. This figure clearly demonstrates a strong strong comovement between the non-commodity expected global inflation component and LR-inflation factor. The correlation between the non-commodity component and LR-inflation factor is 0.695, whereas that between the commodity price-driven component and LR-inflation factor is only 0.196. Therefore, the LR-inflation factor seems to capture the portion of the expected global inflation that is not explained by the commodity sectors.

3.2.3 The SR-inflation Factor

In our theoretical open economy model, the short-term rate is determined by the Taylor rule(equation (8)). Combining the Taylor rule and the equation (4) yields the following relationship between short-term rate and global inflation.

$$S_t = s^* + \rho_1[(1-\alpha)\mathbb{E}_t[\pi^H_{t+q}] + \alpha\mathbb{E}_t[\pi^W_{t+q}] + \alpha\mathbb{E}_t[\Delta e_{t+q}] - \pi^*] + \rho_2\widetilde{y}_t + u_t$$

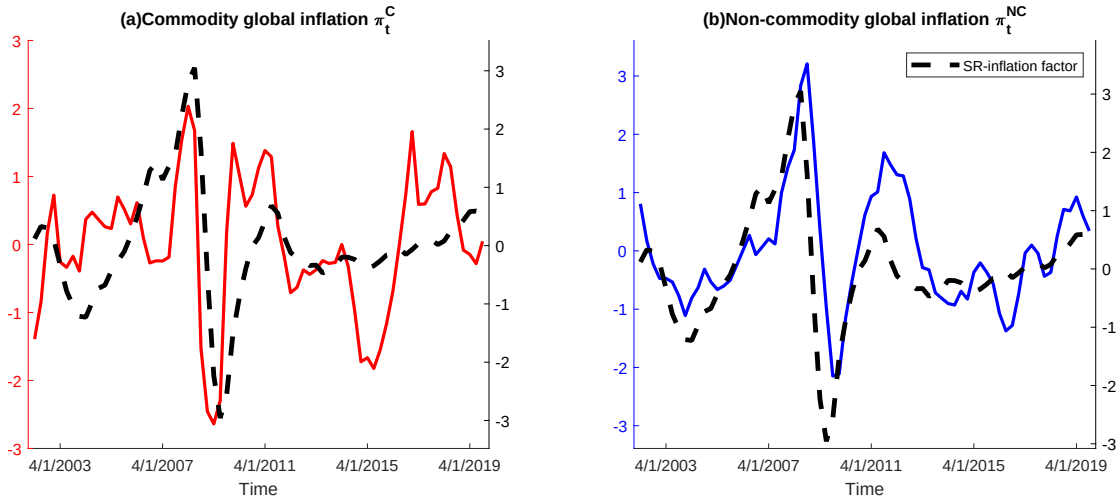
As discussed in section 3.2.2, the global inflation rate can be divided into the commodity price-driven component and the non-commodity component. Thus, we can rewrite the Taylor rule of each country as

$$S_t = s^* + \rho_1(1-\alpha)E_t[\pi^H_{t+q}] + \rho_1\alpha E_t[\pi^C_{t+q}] + \rho_1\alpha E_t[\pi^{NC}_{t+q}] + \rho_1\alpha E_t[\Delta e_{t+q}] - \rho_1\pi^* + \rho_2\widetilde{y}_t + u_t \tag{14}$$

Here, we also show that the non-commodity global inflation component plays an important role in explaining the cross-country comovements in the short-term interest rates and inflation rates. First of all, Figure 10 demonstrates that the SR-inflation factor dynamics is similar to the non-commodity global inflation estimated in subsection 3.2.2. The correlation between the SR-inflation factor and non-commodity global inflation is 0.661. This is much larger than the correlation between the factor and the commodity price-driven global inflation, 0386.

Responses of the shor-term rate to the expected inflation arise from the forward-

Figure 10: **Economic interpretations of the SR-inflation factor** The dashed black line shows the estimated dynamics of the SR-inflation factor. The solid red line shows the commodity global inflation. The solid blue line represents the non-commodity global inflation.

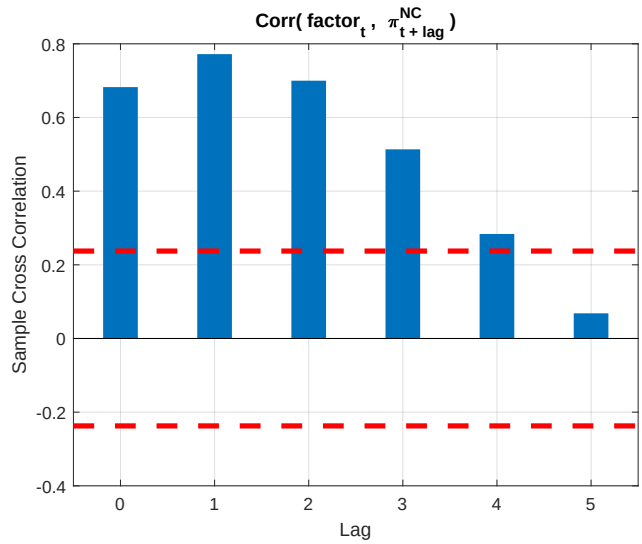


looking nature of policy. Considering the premptive monetary policy, we also estimate the cross-correlation between the global short-rate and non-commodity global inflation using $PC(S)$ as an empirical proxy of the global short-rate. The results are given in Figure 11. The cross-correlation is statistically significant for the lags of 0 to 4. Generally speaking, non-commodity inflation changes are caused by aggregate demand shocks rather than cost shocks. Therefore, our finding is consistent with the conventional wisdom that central banks more aggressively response to demand shocks compared to supply shocks.

4 Conclusion

This paper investigates the common nature of two international synchronization phenomena: the cross-country comovement of the interest rates and the cross-country comovement of inflation rates. By estimating a high-dimensional common factor model, we find two bridges connecting the seeming unrelated comovements. One bridge is the global business cycle which explains the joint dynamics of the national short- and long-term interest rates and inflation rates. The other bridge is the global inflation component

Figure 11: **Cross-correlation between the SR-inflation factor Sπ_t and non-commodity global inflation** π_t^{NC} denotes the non-commodity global inflation. The red dashed horizontal line represents a 95% confidence band.



that is unrelated to the global commodity price. This channel plays an important role in linking the short-term interest rates and inflation rates or the long-term interest rates and inflation rates. We believe that our work helps understand the joint dynamics of international interest rates and inflation rates. On the other hand, our approach is purely statistical. Given the empirical evidence, it would be interesting to set up a general equilibrium model, in which the structural shocks and the causal channels to explain the international comovements can be identified.

References

Altansukh, G., Becker, R., Bratsiotis, G., and Osborn, D. R. (2017), "What is the Globalisation of Inflation?" *Journal of Economic Dynamics and Control*, 74, 1–27.

Carter, C. K. and Kohn, R. (1994), "On gibbs sampling for state space models," *Biometrika*, 81, 541–553.

Chin, M., Graeve, F. D., Filippeli, T., and Theodoridis, K. (2018), "Understanding

International Long-Term Interest Rate Comovement," Tech. rep., Cardiff University, Cardiff Business School, Economics Section.

Ciccarelli, M. and Mojon, B. (2010), "Global inflation," *The Review of Economics and Statistics*, 92, 524–535.

Clarida, R., Gali, J., and Gertler, M. (2000), "Monetary policy rules and macroeconomic stability: evidence and some theory," *The Quarterly journal of economics*, 115, 147–180.

Dahlquist, M. and Hasseltoft, H. (2013), "International Bond Risk Premia," *Journal of International Economics*, 90, 17 – 32.

Del Negro, M., Giannone, D., Giannoni, M. P., and Tambalotti, A. (2018), "Global trends in interest rates," *Working Paper*, 118, 248–262.

Diebold, F. X., Li, C., and Yue, V. Z. (2008), "Global yield curve dynamics and interactions: a dynamic Nelson–Siegel approach," *Journal of Econometrics*, 146, 351–363.

Galí, J. and Monacelli, T. (2005), "Monetary Policy and Exchange Rate Volatility in a Small Open Economy," *The Review of Economic Studies*, 72, 707–734.

Gelman, A., Goodrich, B., Gabry, J., and Vehtari, A. (2019), "R-squared for Bayesian Regression Models," *The American Statistician*, 73, 307–309.

Ha, J., Kose, A., and Ohnsorge, F. L. (2019), *Global inflation synchronization*, The World Bank.

Jotikasthira, C., Le, A., and Lundblad, C. (2015), "Why do term structures in different currencies co-move?" *Journal of Financial Economics*, 115, 58–83.

Kim, C.-J. and Nelson, C. R. (1999), "State-space models with regime switching: classical and Gibbs-sampling approaches with applications," *MIT Press Books*, 1.

Mikolajun, I. and Lodge, D. (2016), "Advanced economy inflation: the role of global factors," Tech. rep., ECB Working Paper.

Monacelli, T. and Sala, L. (2009), "The international dimension of inflation: evidence from disaggregated consumer price data," *Journal of Money, Credit and Banking*, 41, 101–120.

Parker, M. (2017), "Global inflation: the role of food, housing and energy prices," *Working Paper*.

Sutton, G. D. (2000), "Is there excess comovement of bond yields between countries?" *Journal of International Money and Finance*, 19, 363–376.

Takáts, E. and Vela, A. (2014), "International monetary policy transmission," *BIS paper*.

Taylor, J. B. (1993), "Discretion versus policy rules in practice," *Carnegie-Rochester Conference Series on Public Policy*, 39, 195 – 214.

Appendix

A Posterior inference: Gibbs-sampling

For simplicity, Ξ denotes the parameter set except for the latent factors. Also, $\Xi_{-\alpha}$ denotes a set which contains all parameter in Ξ except for α. The appendix describes the complete conditional posterior distribution for each parameter block.

A.1 Transition equation parameters

All latent factors, including the common factors and the country-specific factors, are assumed to follow an independent AR(1) process. Therefore, for each latent factor, AR(1) slope coefficient and error variance can be estimated independently. The AR(1) slope coefficients are sampled using the Normal-Normal update. The error variances are sampled by the InverseGamma-InverseGamma update.

For example, the dynamics of $LS\pi_t$ can be written as follows.

$$LS\pi_t = \phi LS\pi_{t-1} + v_t$$
$$v_t \backsim \mathcal{N}(0, \sigma^2_{F,1})$$

where, ϕ is a AR(1) coefficient, and $\sigma^2_{F,1}$ is a error variance.

In this case, for the dependent variable $Y = \begin{bmatrix} LS\pi_1 & LS\pi_2 & \cdots & LS\pi_T \end{bmatrix}'$ and the independent variable $X = \begin{bmatrix} LS\pi_0 & LS\pi_1 & \cdots & LS\pi_{T-1} \end{bmatrix}'$, the complete conditional posterior distributions of ϕ and $\sigma^2_{F,1}$ are derived as follows.

$$\begin{aligned}
\text{Prior:}&\phi \backsim \mathcal{N}(\phi_0, \Phi_0) \\
&:\sigma^2_{F,1} \backsim \mathcal{IG}(\frac{\sigma_0}{2}, \frac{\Sigma_0}{2}) \\
\text{Posterior:}&\phi|\Xi_{-\phi}, Y, X \backsim \mathcal{N}(\phi_1, \Phi_1) \\
&:\sigma^2_{F,1}|\Xi_{-\sigma^2_{F,1}}, Y, X \backsim \mathcal{IG}(\frac{\sigma_1}{2}, \frac{\Sigma_1}{2}) \\
\text{with, } &\Phi_1 = (\sigma^{-2}_{F,1} X'X + \Phi_0^{-1})^{-1}
\end{aligned}$$

$$\phi_1 = \Phi_1(\sigma_{F,1}^{-2} X'Y + \Phi_0^{-1}\phi_0)$$
$$\sigma_1 = \sigma_0 + T$$
$$\Sigma_1 = \Sigma_0 + (Y - X\phi)'(Y - X\phi)$$

The parameters of other latent factors can be estimated in the same way.

A.2 Factor Loadings in the measurement equation

The measurement equation is $y_t = VF_t + \epsilon_t$, and the transition equation of country-specific factors is $\epsilon_t = \Theta\epsilon_{t-1} + e_t$, where $e_t \backsim \mathcal{N}(O_{3N\times 1}, \Omega_\epsilon)$. Combining these two equations leads to the below equation.

$$y_t - \Theta y_{t-1} = VF_t - \Theta VF_{t-1} + e_t \tag{15}$$

Since Ω_ϵ is a diagonal matrix, we do not need to estimate all measurement equations simultaneously. In other words, we estimate each univariate measurement equation independently. For example, the measurement equation of first country's long-term yield can be written as follows.

$$L_{1,t} = \alpha_1^G LS\pi_t + \alpha_1^Y LS_t + \alpha_1^L L\pi_t + \alpha_1^* L_t^* + \epsilon_{1,t}$$

where, $L_{1,t}, \alpha_1^G, \alpha_1^Y, \alpha_1^L, \alpha_1^*$,and $\epsilon_{1,t}$ represent the first element of $L_t, \alpha^G, \alpha^Y, \alpha^L, \alpha^*$,and ϵ_t, respectively. As in eq(15), the above equation can be transformed to

$$\begin{aligned} L_{1,t} - \theta_1 L_{1,t-1} = {} & \alpha_1^G(LS\pi_t - \theta_1 LS\pi_{t-1}) + \alpha_1^Y(LS_t - \theta_1 LS_{t-1}) \\ & + \alpha_1^L(L\pi_t - \theta_1 L\pi_{t-1}) + \alpha_1^*(L_t^* - \theta_1 L_{t-1}^*) + e_{1,t} \end{aligned}$$

where, θ_1 is the first diagonal element of Θ, $e_{1,t} = \epsilon_{1,t} - \theta_1\epsilon_{1,t-1} \backsim \mathcal{N}(0, \sigma_{\epsilon,1}^2)$, and $\sigma_{\epsilon,1}^2$ is the first diagonal element of Ω_ϵ.

We derive the complete conditional posterior distribution using the Normal-Normal update. For a dependent variable

$$Y = \begin{bmatrix} L_{1,2} - \theta_1 L_{1,1} & L_{1,3} - \theta_1 L_{1,2} & \cdots & L_{1,T}^* - \theta_1 L_{1,T-1}^* \end{bmatrix}'$$

and independent variables

$$X = \begin{pmatrix} LS\pi_2 - \theta_1 LS\pi_1 & LS_2 - \theta_1 LS_1 & L\pi_2 - \theta_1 L\pi_1 & L_2^* - \theta_1 L_1^* \\ LS\pi_3 - \theta_1 LS\pi_2 & LS_3 - \theta_1 LS_2 & L\pi_3 - \theta_1 L\pi_2 & L_3^* - \theta_1 L_2^* \\ \vdots & \vdots & \vdots & \vdots \\ LS\pi_T - \theta_1 LS\pi_{T-1} & LS_T - \theta_1 LS_{T-1} & L\pi_T - \theta_1 L\pi_{T-1} & L_T^* - \theta_1 L_{T-1}^* \end{pmatrix}$$

the posterior distribution of $\alpha = \begin{bmatrix} \alpha_1^G & \alpha_1^Y & \alpha_1^L & \alpha_1^* \end{bmatrix}'$ is as follows.

$$\text{Prior:}\alpha \backsim \mathcal{N}(\alpha_0, A_0)$$

$$\text{Posterior:}\alpha | \Xi_{-\alpha}, Y, X \backsim \mathcal{N}(\alpha_1, A_1)$$

$$A_1 = (\sigma_{\epsilon,1}^{-2} X'X + A_0^{-1})^{-1}$$

$$\alpha_1 = A_1(\sigma_{\epsilon,1}^{-2} X'Y + A_0^{-1}\alpha_0)$$

Other univariate measurement equations can be estimated in the same way.

A.3 Latent variables F_t and ϵ_t

By eq (15), the measurement equation can be written as follows.

$$\underbrace{y_t - \Theta y_{t-1}}_{Y_t} = \underbrace{\begin{bmatrix} V & -\Theta V \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} F_t \\ F_{t-1} \end{bmatrix}}_{\beta_t} + e_t$$

$$e_t \backsim \mathcal{N}(O_{3N\times 1}, \Omega_\epsilon)$$

Using the transition equation of F_t, the transition equation of β_t is as follows.

$$\underbrace{\begin{bmatrix} F_t \\ F_{t-1} \end{bmatrix}}_{\beta_t} = \underbrace{\begin{pmatrix} \Phi & O_{7\times 7} \\ I_7 & O_{7\times 7} \end{pmatrix}}_{G} \underbrace{\begin{bmatrix} F_{t-1} \\ F_{t-2} \end{bmatrix}}_{\beta_{t-1}} + \underbrace{\begin{bmatrix} v_t \\ O_{7\times 1} \end{bmatrix}}_{v_t^o}$$

$$v_t^o \backsim \mathcal{N}(O_{14\times 1}, \Omega_F^o \equiv \begin{pmatrix} \Omega_F & O_{7\times 7} \\ O_{7\times 7} & O_{7\times 7} \end{pmatrix})$$

Therefore, the measurement equation and the transition equation can be re-written as follows.

$$Measurement\ equation : Y_t = H\beta_t + e_t$$

$$Transition\ equation : \beta_t = G\beta_{t-1} + v_t^o$$

To sample latent factors F_t and ϵ_t, we follow Carter and Kohn (1994) and Kim and Nelson (1999). Sampling latent factors can be done in three steps. In the first step, we use the Kalman filter to calculate $\{\beta_{t|t}\}_{t=1}^{T}$ and $\{P_{t|t}\}_{t=1}^{T}$, where $\beta_{i|j}$ is $E[\beta_i|\{Y_\tau\}_{\tau=1}^{j}, \Xi]$, and $P_{i|j}$ is $Var[\beta_i|\{Y_\tau\}_{\tau=1}^{j}, \Xi]$. Then, in the second step, we sample β_t using the backward recursion. Finally, in the third step, we estimate initial latent factors F_0 and ϵ_0 using the Normal-Normal update.

A.3.1 Step 1. Kalman filter

In this step, we derive the distribution $\beta_t|\{Y_\tau\}_{\tau=1}^{t}, \Xi$. We can derive the below equations using properties of the normal distribution.

$$
\begin{aligned}
\beta_{t|t-1} &= G\beta_{t-1|t-1} \\
P_{t|t-1} &= GP_{t-1|t-1}G' + \Omega_F^o \\
Y_{t|t-1} &\equiv E[Y_t|\{Y_\tau\}_{\tau=1}^{t-1}, \Xi] = H\beta_{t|t-1} \\
f_{t|t-1} &\equiv Var[Y_t|\{Y_\tau\}_{\tau=1}^{t-1}, \Xi] = HP_{t|t-1}H' + \Omega_\epsilon \\
\beta_{t|t} &= \beta_{t|t-1} + P_{t|t-1}H'(f_{t|t-1})^{-1}(Y_t - Y_{t|t-1}) \\
P_{t|t} &= P_{t|t-1} - P_{t|t-1}H'(f_{t|t-1})^{-1}HP_{t|t-1}
\end{aligned}
$$

Our Kalman filter algorithm is initiated with the below starting point.

$$
\begin{aligned}
\beta_{1|1} &= \begin{bmatrix} O_{1\times 7} & F_0' \end{bmatrix}' \\
P_{1|1} &= \begin{pmatrix} \Omega_F^{un} & O_{7\times 7} \\ O_{7\times 7} & O_{7\times 7} \end{pmatrix}
\end{aligned}
$$

where, Ω_F^{un} is the unconditional variance of F_t. With the above initial point, we can start the algorithm to calculate $\{\beta_{t|t}\}_{t=1}^{T}, \{P_{t|t}\}_{t=1}^{T}$ recursively.

A.3.2 Step 2. Backward recursion

The backward recursion is initialized with F_T, where F_T is in β_T, and β_T can be sampled from $\beta_T|\{Y_\tau\}_{\tau=1}^{T}, \Xi$. And then, for the recursion, we use the below relationship derived

from the transition equation.

$$F_t = \underbrace{\begin{bmatrix}\Phi & O_{7\times 7}\end{bmatrix}}_{G^*}\underbrace{\begin{bmatrix}F_{t-1}\\ F_{t-2}\end{bmatrix}}_{\beta_{t-1}} + v_t$$

Given F_{t+1}, we sample β_t from $\beta_t|\{Y_\tau\}_{\tau=1}^t, F_{t+1}$, and we only store F_t in β_t. This backward recursion continues until F_1 is sampled. $\beta_t|\{Y_\tau\}_{\tau=1}^t, F_{t+1}$ is a normal distribution, and the distribution's expectation and variance are as follows.

$$E[\beta_t|\{Y_\tau\}_{\tau=1}^t, F_{t+1}] = \beta_{t|t} + P_{t|t}G^{*\prime}(G^*P_{t|t}G^{*\prime} + \Omega_F)^{-1}(F_{t+1} - G^*\beta_{t|t})$$
$$Var[\beta_t|\{Y_\tau\}_{\tau=1}^t, F_{t+1}] = P_{t|t} - P_{t|t}G^{*\prime}(G^*P_{t|t}G^{*\prime} + \Omega_F)^{-1}G^*P_{t|t}$$

With $\{F_t\}_{t=1}^T$ in hand, $\{\epsilon_t\}_{t=1}^T$ can be calculated by the below measurement equation.

$$\epsilon_t = y_t - VF_t$$

A.3.3 Step 3. Initial latent factor sampling

The measurement equation and the transition equation can be re-written as follows.

$$y_t = \underbrace{\begin{bmatrix}V & I_{3N}\end{bmatrix}}_{\Gamma}\underbrace{\begin{bmatrix}F_t\\ \epsilon_t\end{bmatrix}}_{Z_t}$$
$$\underbrace{\begin{bmatrix}F_t\\ \epsilon_t\end{bmatrix}}_{Z_t} = \underbrace{\begin{pmatrix}\Phi & O_{7\times 3N}\\ O_{3N\times 7} & \Theta\end{pmatrix}}_{G_Z}\underbrace{\begin{bmatrix}F_{t-1}\\ \epsilon_{t-1}\end{bmatrix}}_{Z_{t-1}} + \underbrace{\begin{bmatrix}v_t\\ e_t\end{bmatrix}}_{z_t}$$
$$\begin{bmatrix}v_t\\ e_t\end{bmatrix} \backsim \mathcal{N}(O_{(7+3N)\times 1}, \Omega_Z \equiv \begin{pmatrix}\Omega_F & O_{7\times 3N}\\ O_{3N\times 7} & \Omega_\epsilon\end{pmatrix})$$

Combining the above equations yield the following equation.

$$y_1 = \Gamma G_Z Z_0 + \Gamma z_1$$
$$\Gamma z_1 \backsim \mathcal{N}(O_{3N\times 1}, \Gamma\Omega_Z\Gamma')$$

With Cholesky decomposition $\mathcal{L}\mathcal{L}'$=$\Gamma\Omega_Z\Gamma'$, the above equation can be transformed to the below equation.

$$\mathcal{L}^{-1}y_1 = \mathcal{L}^{-1}\Gamma G_Z Z_0 + \mathcal{L}^{-1}\Gamma z_1$$
$$\mathcal{L}^{-1}\Gamma z_1 \backsim \mathcal{N}(O_{3N\times 1}, I_{3N})$$

Therefore, Z_0 can be sampled from the Normal-Normal update with a dependent variable matrix $Y \equiv \mathcal{L}^{-1}y_1$ and an independent matrix $X \equiv \mathcal{L}^{-1}\Gamma G_Z$.

$$\text{Prior:} Z_0 \backsim \mathcal{N}(0, \Omega_Z^{un})$$
$$\text{Posterior:} Z_0|\Xi, Y, X \backsim \mathcal{N}(Z^*, \Omega_Z^*)$$
$$\Omega_Z^* = (X'X + (\Omega_Z^{un})^{-1})^{-1}$$
$$Z^* = \Omega_Z^* X'Y$$

where, Ω_Z^{un} is an unconditional variance matrix of Z_t, derived from the transition equation of F_t and ϵ_t.