Equilibrium with Information Acquisition under Ambiguity

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Abstract

This paper analyzes a model of a competitive asset market in which all the traders have ambiguous information at the beginning and can choose whether to resolve ambiguity by acquiring costly information or to remain uninformed with ambiguity. We find that risk-averse traders consider the opposite probability distribution depending on their information choice and this leads to multiple equilibria and strategic complementarities in information acquisition.

KEYWORDS: ambiguity; asymmetric information; information acquisition; strategic complementarity; multiple equilibria

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1 Introduction

Information plays a key role in determining traders' choices in financial markets. Investors always try to collect more information about future values of assets from various sources for making better investment decisions. According to their status or experiences, some traders could have only poor information about them, while others easily access to core information about future payoffs. Indeed, some traders depend on public information such as earnings reports, daily news, current asset prices, and many others, while other traders use private inside information to make portfolio choices. Practically, some information obtained from doubtful sources may make traders pose questions about its accuracy. They may have multiple prior beliefs about distributions of assets' future values.

In Grossman and Stiglitz (1980), as more traders purchase information, it becomes easier for uninformed traders to infer informed traders' information by observing asset prices. This leads to the decrease of the value of information. However, recent literature shows that asset market equilibrium can exhibit strategic complementarity in information acquisition by various reasons. Barlevy and Veronesi (2000) and Chamley (2007) claim that strategic complementarity in information acquisition is observed in models without CARA-normal specification. They show the existence of the strategic complementarity by assuming risk neutrality and binomial distribution.[1] In particular, Chamley (2007) shows that strategic complementarity occurs due to low confidence level of public beliefs about an asset's future payoffs. According to Hellwig and Veldkamp (2009), strategic feature of information choice depends on actions of traders. In fact, information choice becomes strategic complement when traders play a game with strategic complementarity and becomes strategic substitute otherwise. Other studies show that strategic complementarity can occur under the standard CARA-normal assumptions. Among others, Rahi and Zigrand (2015) show that strategic complementarity is observed if traders obtain heterogenous information and Mele and Sangiorgi (2015) show that it arises due to ambiguity about the mean.

This paper shows how ambiguity generates strategic complementarity in information acquisition. Facing ambiguity, would-be informed traders and would-be uninformed traders consider different probability distributions when they maximize their ex ante expected utility. For would-be informed traders, minimum variance of the true value is their worst-case distribution since it makes the quality of information become the lowest. On the other hand, would-be uninformed traders consider the maximum variance of the true value as their worst-case distribution. This difference leads to the possibility of multiple equilibria. We find that if the cost of information lies in some range, both equilibria with strategic substitutability and with strategic complementarity coexist in information markets. Equilibria with strategic complementarity occurs when the proportion of informed traders is sufficiently low. These equilibria is unstable since when a trader purchases

[1] Chamley (2008) claims that there is computation error in Barlevy and Veronesi (2000) and strategic complementarity is not observed in their model.

information, the other traders have incentives to be informed. It is shown that the increase of information cost leads to the decrease of the proportion of informed traders in equilibrium with strategic substitutability and leads to the reverse effect in equilibrium with strategic complementarity.

We also examine the effects of ambiguity on information markets. In the equilibria with strategic substitutability, as the minimum variance decreases, the proportion of informed traders decreases, while as the maximum variance increases, the proportion of informed traders increases. In the equilibria with strategic complementarity, changes of the minimum variance and the maximum variance reversely affect the proportion of informed traders.

This paper is closely related to Mele and Sangiorgi (2015) in that both adopt the framework of Grossman and Stiglitz (1980) and attempt to explain strategic complementarity in information acquisition by using ambiguity. However, this paper is distinct from Mele and Sangiorgi (2015) in two aspects. First, we assume ambiguity about the variance while they assume that about the mean. Thus portfolio inertia does not arise in our model and thus we obtain a linear price function. This makes our analysis simpler. Second, our model elucidates the reason why strategic complementarity in information acquisition occurs under ambiguity. Mele and Sangiorgi (2015) only show the existence of the strategic complementarity. We verify that the difference in the worst-case probability distribution between would-be informed and uninformed traders can induce strategic complementarity.

The rest of the paper is organized as follows. In Section 2, we introduce the model of asset markets under asymmetric information with ambiguity. Asset price under ambiguity is found in Section 3. Section 4 analyzes the characteristics of the equilibrium in the ex ante stage. Concluding remarks are given in Section 5.

2 The Model

The model has three periods, $0, 1$, and 2. There are two assets: a risky asset and a risk-free bond. The (risky) asset gives random payoff $\tilde{v}$, which is the sum of true value $\tilde{\theta}$ and noise $\tilde{\varepsilon}$: $\tilde{v} = \tilde{\theta} + \tilde{\varepsilon}$. True value $\tilde{\theta}$ and noise $\tilde{\varepsilon}$ are normally distributed with means μ and 0 and variances σ^2 and σ_ε^2, respectively. We assume that the volatility of the true value is higher than that of noise, i.e., $\sigma^2 > \sigma_\varepsilon^2$. The prices of the asset and the bond are given by p and 1 at period 1, respectively. At period 2, payoff v of the asset is realized and the bond gives payoff 1. We assume that asset supply $\tilde{z}$ is also a random variable, which is normally distributed with mean zero and variance σ_z^2. All random variables are mutually independent.

All the rational traders are identical at period 0. They have ambiguous information about σ^2 with knowing that it belongs to $[\underline{\sigma}^2, \bar{\sigma}^2]$ but have exact information about μ. We call $\underline{\sigma}^2$ ($\bar{\sigma}^2$) the minimum (maximum, respectively) variance (of the fundamental value). In this period, rational traders should decide whether to purchase information about θ at cost c or not. We call rational

traders who have information about θ informed traders and the other rational traders uninformed traders. Let $\lambda \in [0,1]$ denote the fractions of informed traders. At period 1, in the interim stage, informed traders observe realization (p,θ) of $(\tilde{p},\tilde{\theta})$, while uninformed traders and only observe p. After knowing true value θ of the asset, informed traders resolve ambiguity when they make portfolio choices. On the other hand, uninformed traders are still under ambiguity. All the traders have rational expectations so that they understand the functional relationship $\tilde{p}$ between p and (θ, z). We illustrate the sequence of events of the model in Figure 1.

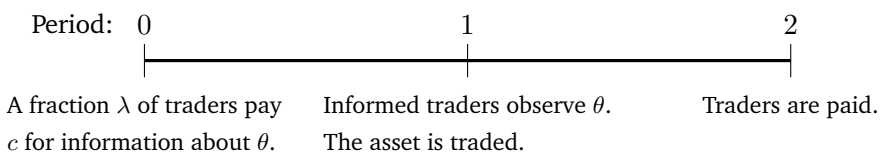

Figure 1: Sequence of events

A trader t invests his initial wealth w_t between x_t shares of the asset and b_t shares of the bond with the budget constraint $b_t + px_t = w_t$ at period 1. Thus his portfolio (b_t, x_t) yields wealth $w_t' = w_t + (\tilde{v} - p)x_t$ at period 2. Suppose that all the traders have CARA utility function with the constant degree of risk aversion $\gamma > 0$: $u(w) = -\exp(-\gamma w)$.

For the optimal portfolio choice, informed trader i with initial wealth w_0 solves

$$\max_{\hat{x}_i} \mathbb{E}[-\exp\left(-\gamma[w_0 - c + (\tilde{v} - p)\hat{x}_i]\right) | (\tilde{p},\tilde{\theta}) = (p,\theta)]$$

and his demand for the asset is given by

$$x_i(p,\theta) = \frac{\theta - p}{\gamma\sigma_\varepsilon^2}. \tag{2.1}$$

Let $\mathcal{P}$ be the set of normal distributions with mean μ and variance $\hat{\sigma}^2 \in [\underline{\sigma}^2, \bar{\sigma}^2]$. The mean and variance under belief $\pi \in \mathcal{P}$ are denoted by $\mathbb{E}_\pi[\cdot]$ and $\text{Var}_\pi[\cdot]$, respectively. Equipped with MEU preferences of Gilboa and Schmeidler (1989), uninformed trader u with initial wealth w_0 solves

$$\max_{\hat{x}_u} \min_{\pi\in\mathcal{P}} \mathbb{E}_\pi\left[-\exp\left(-\gamma[w_0 + (\tilde{v} - p)\hat{x}_u]\right) | \, \tilde{p} = p\right],$$

or equivalently,

$$\max_{\hat{x}_u} \min_{\pi\in\mathcal{P}} \left[\gamma\left(w_0 + \hat{x}_u(\mathbb{E}_\pi[\tilde{v}|\tilde{p} = p] - p) - \frac{1}{2}\gamma\hat{x}_u^2 \text{Var}_\pi[\tilde{v}|\tilde{p} = p]\right)\right]. \tag{2.2}$$

The multiplicity of beliefs about σ^2 posed by a uninformed trader affects both conditional expectation and variance of liquidation value v given stock price p.[2] Thus uninformed trader's worst-case

[2] As in Mele and Sangiorgi (2015), ambiguity on μ only affects the expected value of v conditional on p.

scenario for the expectation and variance of v cannot be dealt with separately and takes into account simultaneous effects of probability distribution he may consider on the mean and variance. From (2.2), we notice that a uninformed trader who would take a long (short) position considers a probability distribution which minimizes (maximizes, respectively) $\mathbb{E}_\pi[\tilde{v}|\tilde{p}=p]$ and maximizes $\text{Var}_\pi[\tilde{v}|\tilde{p}=p]$ as the worst-case scenario. On the other hand, the uninformed trader does not concern the probability distribution when takes zero position.

Let $\hat{\pi} \in \mathcal{P}$ be the probability distribution which minimizes $\mathbb{E}_\pi[\tilde{v}|\tilde{p}=p]$ and maximizes $\text{Var}_\pi[\tilde{v}|\tilde{p}=p]$, and $\bar{\pi} \in \mathcal{P}$ be the distribution which maximizes both $\mathbb{E}_\pi[v|\tilde{p}=p]$ and $\text{Var}_\pi[\tilde{v}|\tilde{p}=p]$. Provided that $\hat{\pi}$ and $\bar{\pi}$ exist, the demand of uninformed trader u for the asset is given by

$$x_u(\tilde{p}) = \begin{cases} \dfrac{\mathbb{E}_{\hat{\pi}}[\tilde{v}|\tilde{p}=p]-p}{\gamma \text{Var}_{\hat{\pi}}[\tilde{v}|\tilde{p}=p]}, & \text{if} \quad p < \mathbb{E}_{\hat{\pi}}[\tilde{v}|\tilde{p}=p], \\ 0, & \text{if} \quad \mathbb{E}_{\hat{\pi}}[\tilde{v}|\tilde{p}=p] \le p \le \mathbb{E}_{\bar{\pi}}[\tilde{v}|\tilde{p}=p], \\ \dfrac{\mathbb{E}_{\bar{\pi}}[\tilde{v}|\tilde{p}=p]-p}{\gamma \text{Var}_{\bar{\pi}}[\tilde{v}|\tilde{p}=p]}, & \text{if} \quad p > \mathbb{E}_{\bar{\pi}}[\tilde{v}|\tilde{p}=p]. \end{cases} \tag{2.3}$$

One may question the existence of probability distributions $\hat{\pi}$ and $\bar{\pi}$. We verify the existence of such distributions in Section 3.

In the models with ambiguity about mean such as Cao et al. (2005) and Mele and Sangiorgi (2015), uninformed traders take into account different probability distributions depending on observed prices as the worst-case scenarios and further, they refuse to take positions when asset prices fall in some intermediate regions. This implies that portfolio inertia arises and uninformed traders ask ambiguity premium to take positions. Departing from their result, we verify that distributions $\hat{\pi}$ and $\bar{\pi}$ are same, implying $\mathbb{E}_{\hat{\pi}}[\tilde{v}|\tilde{p}=p] = \mathbb{E}_{\bar{\pi}}[\tilde{v}|\tilde{p}=p]$ and $\text{Var}_{\hat{\pi}}[\tilde{v}|\tilde{p}=p] = \text{Var}_{\bar{\pi}}[\tilde{v}|\tilde{p}=p]$ in Section 3. As a result, the demand function becomes linear for every observed price without the non-participation region. This property makes our model more tractable when we analyze information acquisition at the ex ante stage in Section 4.

Unlike the model of Mele and Sangiorgi (2015), we find the probability distributions considered by would-be informed and uninformed traders, which give them the minimum ex ante expected utilities. Since the would-be uninformed traders take the maximum variance while would-be informed one the minimum variance, there exists strategic complementary in information acquisition. We discuss about this in detail in Section 4.

3 Asset Market Equilibrium

In this section, we consider a rational expectations equilibrium in which the proportion λ of informed traders are exogenously given. A rational expectations equilibrium asset price function $\tilde{p} : (\theta, z) \mapsto p$ satisfies the market clearing condition: for every $p = \tilde{p}(\theta, z)$,

$$\lambda x_i(p, \theta) + (1-\lambda) x_u(p, \tilde{p}) = z. \tag{3.1}$$

Following Grossman and Stigltiz (1980),[3] we define a compound signal function $\tilde{s} : (\theta, z) \mapsto s$ for $\lambda \in [0, 1]$, which encapsulates θ and z:

$$\tilde{s}(\theta, z) \equiv \begin{cases} \theta - \frac{\gamma\sigma_\varepsilon^2}{\lambda} z & \text{if } \lambda \in (0, 1], \\ -z & \text{if } \lambda = 0. \end{cases}$$

Clearly, $\tilde{s}$ is normally distributed with mean μ and variance $\sigma_s^2 = \sigma^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2/\lambda^2$ if $\lambda \in (0, 1]$ and with mean zero and variance σ_z^2 if $\lambda = 0$. We define equilibrium asset price function $P : s \mapsto p$ by $P(\tilde{s}(\theta, z)) := \tilde{p}(\theta, z)$ and conjecture that P strictly increases in signal s, which is verified by Proposition 3.1 below.

Proposition 3.1. *For every $\lambda \in [0, 1]$, there exists a unique rational expectations equilibrium asset price function P, given by*[4]

$$P(s) = (1 - \alpha)\mu + \alpha s, \quad \forall\, s \in \mathbb{R} \tag{3.2}$$

where

$$\alpha \equiv \frac{\lambda(\lambda\bar{\sigma}^2 + \gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2)}{\lambda^2\bar{\sigma}^2 + \lambda\gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2}.$$

Proposition 3.1 shows that the equilibrium asset price is a linearly increasing function of s. Observing signal s, uninformed traders extract equivalent information about the asset's fundamental value to which they could obtain from asset prices. It is notable that uninformed traders refuse to take positions until observed asset prices are sufficiently low or high.

After observing the signal, ambiguity-averse uninformed traders take the probability distribution with the maximum variance $\bar{\sigma}^2$ as the worse case scenario whether they take long or short positions. On the other hand, in the models of Ozsoylev and Werner (2011) and Mele and Sangiorgi (2015) which assume ambiguity on mean, uniformed traders allow for different probability distributions depending on their positions.

Since

$$\frac{\partial\alpha}{\partial\bar{\sigma}^2} = \frac{(1 - \lambda)(\lambda + \gamma^2\sigma_\varepsilon^2\sigma_z^2)\lambda\gamma^2\sigma_\varepsilon^4\sigma_z^2}{(\lambda^2\bar{\sigma}^2 + \lambda\gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2)^2} \geq 0,$$

as the maximum variance increases, stock price P response to s more sensitively.

Corollary 3.1. *The demand function $x_u : \mathbb{R} \to \mathbb{R}$ of uninformed traders is given by*

$$x_u(s) = \kappa(\mu - s). \tag{3.3}$$

where

$$\kappa \equiv \frac{\lambda\gamma\sigma_\varepsilon^2\sigma_z^2}{\lambda^2\bar{\sigma}^2 + \lambda\gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2}.$$

[3] Mele and Sangiorgi (2015) use a slightly different signal function from ours and Grossman and Stigltiz (1980).

[4] Note that the equilibrium price function is continuous at $\lambda = 0$, we can express the equilibrium function by a single equation although $\tilde{s}$ has different from when $\lambda = 0$ and $\lambda \in (0, 1]$.

Note that (3.3) is just another expression of (2.3). If observed stock prices are sufficiently low such that $s < \mu$ or sufficiently high such that $s > \mu$, uninformed traders' expected utility in the interim stage is minimized when they take the maximum variance (i.e., $\sigma^2_{\hat{\pi}} = \sigma^2_{\bar{\pi}} = \bar{\sigma}^2$). In other words, uninformed traders consider the same probability distribution with variance $\bar{\sigma}^2$ whenever they take positions. Since distributions $\hat{\pi}$ and $\bar{\pi}$ are equivalent, the size of non-participation region $[\mathbb{E}_{\hat{\pi}}[\tilde{v}|\tilde{p} = p], \mathbb{E}_{\bar{\pi}}[\tilde{v}|\tilde{p} = p]]$ becomes zero. It follows that the uniformed traders' demand function is linear with $s \in \mathbb{R}$.[5]

Clearly, as maximum variance $\bar{\sigma}^2$ increases, κ decreases. Thus, uninformed traders respond to price signals less aggressively as they face a higher maximum variance. On the other hand, minimum variance $\underline{\sigma}^2$ does not affect the uninformed traders' demand choice.

4 Information Acquisition

In this section, we derive overall equilibrium of the asset market. Facing ambiguity, at the ex ante stage, rational traders choose whether they are informed or not. We find information market equilibrium at the proportion of informed traders where rational traders have the equivalent ex ante expected utility. Finding the equilibrium λ, we put this into the equilibrium price derived in Section 3.

4.1 Information Market Equilibrium

At period 0, all the traders have ambiguous information about the variance σ^2 of the true value θ. Taking probability distribution which minimizes their ex ante expected utility, rational traders choose whether to become informed at cost c.

Proposition 4.1. *The following hold.*

1. *The ex ante expected utility of would-be informed trader i is given by*

$$U_i(\lambda; c) = \min_{\sigma^2_{\pi_i} \in [\underline{\sigma}^2, \bar{\sigma}^2]} \mathbb{E}[u(w_i)] = e^{\gamma c} u(w_0) \sqrt{h_i(\lambda)}$$

where

$$h_i(\lambda) \equiv \frac{(\lambda^2 \bar{\sigma}^2 + \lambda \gamma^2 \bar{\sigma}^2 \sigma^2_\varepsilon \sigma^2_z + \gamma^2 \sigma^4_\varepsilon \sigma^2_z)^2}{[(1-\lambda)^2 \underline{\sigma}^2 + \gamma^2 \sigma^4_\varepsilon \sigma^2_z + \sigma^2_\varepsilon] \gamma^4 \sigma^6_\varepsilon \sigma^4_z + (\lambda + \gamma^2 \sigma^2_\varepsilon \sigma^2_z)^2 [(\lambda^2 + \gamma^2 \sigma^2_\varepsilon \sigma^2_z) \bar{\sigma}^2 + 2\gamma^2 \sigma^4_\varepsilon \sigma^2_z] \bar{\sigma}^2}.$$

2. *Ex ante expected utility of would-be uninformed traders is given by*

$$U_u(\lambda; c) = \min_{\sigma^2_u \in [\underline{\sigma}^2, \bar{\sigma}^2]} \mathbb{E}_{\sigma^2_u}[u(w_i)] = u(w_0) \sqrt{h_u(\lambda)}$$

[5] Our result is consistent with that of Ozsoylev and Werner (2011), in which ambiguity on both mean and variance is assumed. In their model, whenever taking positions, the uninformed arbitrager takes the maximum variance while considers different means depending on prices.

where

$$h_u(\lambda) \equiv \frac{(\lambda\bar{\sigma}^2 + \lambda\gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2)^2}{X_u(\lambda)}$$

with

$$X_u(\lambda) = \lambda^4\bar{\sigma}^4 + \gamma^2\sigma_\varepsilon^2\sigma_z^2[2\lambda^2\bar{\sigma}^2(\lambda\bar{\sigma}^2+\sigma_\varepsilon^2) + \gamma^2\sigma_\varepsilon^2\sigma_z^2\{\lambda^2\bar{\sigma}^4 + \sigma_\varepsilon^2((2+\lambda)\lambda\bar{\sigma}^2 + \gamma^2\sigma_\varepsilon^2\sigma_z^2(\bar{\sigma}^2+\sigma_\varepsilon^2)+\sigma_\varepsilon^2)\}].$$

Let us define a function of $\lambda \in [0,1]$ given by

$$\xi(\lambda; c) \equiv \frac{U_i(\lambda; c)}{U_u(\lambda; c)} = e^{\gamma c}\sqrt{\frac{h_i(\lambda)}{h_u(\lambda)}}.$$

The function $\xi(\lambda; c)$ represents the ratio of ex ante expected utility of would-be informed traders and would-be uninformed ones. Note that $1/\xi(\lambda; c)$ measures the value of information. Since we have assumed negative utility function, if $\xi(\lambda; c) < 1$, ex ante expected utility of would-be informed traders is higher than that of would-be uninformed ones and if $\xi(\lambda; c) > 1$, ex ante expected utility of would-be uninformed ones is higher than that of would-be informed ones. Note that an increase in information cost c vertically shifts up $\xi(\lambda; c)$. If c is high enough such that $\xi(\lambda; c) > 1$ for every $\lambda \in [0,1]$, ex ante expected utility of would-be uninformed traders is always higher than that of would-be informed ones. Then all traders choose to be uninformed and equilibrium λ becomes zero. On the other hand, if c is sufficiently low such that $\xi(\lambda; c) < 0$ for every $\lambda \in [0,1]$, ex ante expected utility of would-be informed traders is always higher than that of would-be uninformed ones. Then all the traders purchase information at cost c and λ becomes 1 in equilibrium.

Unlike the model of Grossman and Stiglitz (1980), $\xi(\lambda; c)$ is not an increasing function of λ. Indeed, it decreases in λ at $\lambda = 0$ since

$$\frac{\partial}{\partial\lambda}\left(\frac{h_i}{h_u}\right)\bigg|_{\lambda=0} = -\frac{(\bar{\sigma}^2 - \underline{\sigma}^2)(\bar{\sigma}^2 + \sigma_\varepsilon^2)(1 + \gamma^2\sigma_\varepsilon^2\sigma_z^2)}{\sigma_\varepsilon^2(1 + \gamma^2\sigma_z^2(\bar{\sigma}^2 + \sigma_\varepsilon^2))(\underline{\sigma}^2 + \sigma_\varepsilon^2 + \gamma^2\sigma_z^2(\bar{\sigma}^2 + \sigma_\varepsilon^2)^2)} < 0 \tag{4.1}$$

and it increases in λ at $\lambda = 1$ by

$$\frac{\partial}{\partial\lambda}\left(\frac{h_i}{h_u}\right)\bigg|_{\lambda=1} = \frac{\gamma^2\bar{\sigma}^4\sigma_\varepsilon^2\sigma_z^2}{(\bar{\sigma}^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2)(\bar{\sigma}^2 + \gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2)} > 0. \tag{4.2}$$

Here, we can see the possibility of equilibrium with strategic complementarity in information acquisition, at which even though the more traders become informed, the incentive to be informed increases.

Proposition 4.1. *For $\lambda \in [0,1]$, function $\xi(\lambda; c)$ is a convex function.*[6]

Proposition 4.2. *If $\bar{\sigma}^2 > \sigma_\varepsilon^2$, then $\xi(0; c) < \xi(1; c)$.*

[6]The proof is available upon request.

PROOF : We have

$$\frac{h_i(0)}{h_u(0)} = \frac{\sigma_\varepsilon^2(1+\gamma^2\sigma_z^2(\bar{\sigma}^2+\sigma_\varepsilon^2))}{\underline{\sigma}^2+\sigma_\varepsilon^2+\gamma^2\sigma_z^2(\bar{\sigma}^4+2\bar{\sigma}^2\sigma_\varepsilon^2+\sigma_\varepsilon^4)},$$
$$\frac{h_i(1)}{h_u(1)} = \frac{\bar{\sigma}^2+\gamma^2\sigma_\varepsilon^4\sigma_z^2}{\bar{\sigma}^2+\gamma^2\sigma_\varepsilon^2\sigma_z^2(\bar{\sigma}^2+\sigma_\varepsilon^2)},$$

and

$$\frac{h_i(1)}{h_u(1)} - \frac{h_i(0)}{h_u(0)} = \frac{\gamma^2\bar{\sigma}^2\sigma_z^2[\bar{\sigma}^4+\sigma_\varepsilon^2(\bar{\sigma}^2-\sigma_\varepsilon^2)]+\underline{\sigma}^2(\bar{\sigma}^2+\gamma^2\sigma_\varepsilon^4\sigma_z^2)}{[\underline{\sigma}^2+\sigma_\varepsilon^2+\gamma^2\sigma_z^2(\bar{\sigma}^4+2\bar{\sigma}^2\sigma_\varepsilon^2+\sigma_\varepsilon^4)][\bar{\sigma}^2+\gamma^2\sigma_\varepsilon^2\sigma_z^2(\bar{\sigma}^2+\sigma_\varepsilon^2)]},$$

which is greater than zero when $\bar{\sigma}^2 > \sigma_\varepsilon^2$. ■

By (4.1), (4.2) and Proposition 4.1, we obtain the following proposition.

Proposition 4.2. *The following hold.*

1. *There exists unique $\hat{\lambda} \in (0,1)$ which minimizes $\xi(\lambda; c)$.*
2. *The function $\xi(\lambda; c)$ is maximized at $\lambda = 1$.*

Let us define

$$\hat{c} \equiv -\frac{1}{2\gamma}\ln\left(\frac{h_i(\hat{\lambda})}{h_u(\hat{\lambda})}\right),$$
$$\bar{c} \equiv -\frac{1}{2\gamma}\ln\left(\frac{\sigma_\varepsilon^2(1+\gamma^2\sigma_z^2(\bar{\sigma}^2+\sigma_\varepsilon^2))}{\underline{\sigma}^2+\sigma_\varepsilon^2+\gamma^2\sigma_z^2(\bar{\sigma}^4+2\bar{\sigma}^2\sigma_\varepsilon^2+\sigma_\varepsilon^4)}\right),$$
$$\underline{c} \equiv -\frac{1}{2\gamma}\ln\left(\frac{\bar{\sigma}^2+\gamma^2\sigma_\varepsilon^4\sigma_z^2}{\bar{\sigma}^2+\gamma^2\sigma_\varepsilon^2\sigma_z^2(\bar{\sigma}^2+\sigma_\varepsilon^2)}\right).$$

Proposition 4.3. *We have $\hat{c} > \bar{c} > \underline{c}$.*

DEFINITION 4.1.

1. *The information acquisition is strictly strategic substitute if as more traders become informed, the value of information deceases (i.e., $\xi'(\lambda; c) > 0$).*
2. *The information acquisition is strictly strategic complement if as more traders become informed, the value of information increases (i.e., $\xi'(\lambda; c) < 0$).*
3. *The information acquisition is both strategic substitute and complement if a small change in λ does not affect the value of information (i.e., $\xi'(\lambda; c) = 0$).*

Proposition 4.4. *The following hold.*

1. *If information cost c is sufficiently low such that $c \leq \underline{c}$, there exists a unique equilibrium $\lambda^* = 1$ with strategic substitutability.*
2. *If $c \in (\underline{c}, \bar{c})$, there exists a unique equilibrium $\lambda^* \in (0,1)$ with strategic substitutability.*

3. *If $c = \bar{c}$, there exist two equilibria: $\lambda_1^* = 0$ with strategic complementarity and $\lambda_2^* \in (0,1)$ with strategic substitutability.*

4. *If $c \in (\bar{c}, \hat{c})$, then there exist two equilibria: $\lambda_1^* \in (0,1)$ with strategic complementarity and $\lambda_2^* \in (0,1)$ with strategic substitutability where $\lambda_1^* < \lambda_2^*$.*

5. *If $c = \hat{c}$, there exist a unique equilibrium $\lambda^* = \hat{\lambda}$ with strategic substitutability and complementarity.*

6. *If information cost is sufficiently high such that $c > \hat{c}$, there exists a unique equilibrium $\lambda^* = 0$ with strategic complementarity.*

PROOF : Note that an increase in c shifts up ξ. Further, ξ decreases in λ when $\lambda \in (0, \hat{\lambda})$ and increases in λ when $\lambda \in (\hat{\lambda}, 1)$.

(1) By the fact

$$\xi(1; \underline{c}) = e^{\gamma \underline{c}} \sqrt{\frac{h_i(1)}{h_u(1)}} = 1$$

and the second claim of Proposition 4.2, $\xi \leq 1$ for all $\lambda \in [0,1]$. Thus, all rational traders choose to be informed (i.e., $\lambda^* = 1$). Further, by (4.2), the equilibrium is of strategic substitutability.

(2) If $c \in (\underline{c}, \bar{c})$, we have

$$\xi(0; c) < 1 \quad \text{and} \quad \xi(1; c) > 1.$$

Then we have a unique equilibrium $\lambda^* \in (\hat{\lambda}, 1)$. Since $\xi(\lambda; c)$ increases in λ when $\lambda \in (\hat{\lambda}, 1)$, the equilibrium is of strategic substitutability.

(3) At $c = \bar{c}$, we have

$$\xi(0; \bar{c}) = e^{\gamma \bar{c}} \sqrt{\frac{h_i(0)}{h_u(0)}} = 1$$

Since $\xi(0; c) < \xi(1; c)$, there exist two equilibria: $\lambda_1^* = 0$ and $\lambda_2^* \in (\hat{\lambda}, 1)$. Since $\xi(\lambda; c)$ decrease in λ at $\lambda = 0$, λ_1^* is the equilibrium with strategic substitutability. Further, it increases in λ at $\lambda \in (\hat{\lambda}, 1)$, implying that λ_2^* is the equilibrium with strategic complementarity.

(4) If $c \in (\bar{c}, \hat{c})$, we know that $\xi(0; c) > 1$ and $\xi(\hat{\lambda}; c) < 1$. Further, $\xi(1; c) > 1$ since $\xi(1; c) > \xi(0)$. Therefore, there exist two equilibria: $\lambda_1^* \in (0, \hat{\lambda})$ and $\lambda_2^* \in (\hat{\lambda}, 1)$. Since $\xi(\lambda; c)$ decrease in λ when $\lambda \in (0, \hat{\lambda})$, λ_1^* is the equilibrium with strategic complementarity. Since $\xi(\lambda; c)$ increase in λ when $\lambda \in (\hat{\lambda}, 1)$, λ_2^* is the equilibrium with strategic substitutability.

(5) At $c = \hat{c}$, we have $\xi(\hat{\lambda}; c) = 1$. Since ξ uniquely has the minimum value at $\lambda = \hat{\lambda}$, we have the unique equilibrium at $\lambda^* = \hat{\lambda}$. Since the slope of $\xi(\lambda; c)$ is zero and its sign changes at $\hat{\lambda}$, λ^* is the equilibrium with both strategic substitutability and strategic complementarity.

(6) If $c > \hat{c}$, then $\xi(\lambda; c) > 1$ for all $\lambda \in [0,1]$. Thus, all traders choose to be uninformed (i.e., $\lambda^* = 0$). Since $\xi(\lambda; c)$ decrease in λ at $\lambda = 0$, then λ^* is the equilibrium with strategic complementarity.

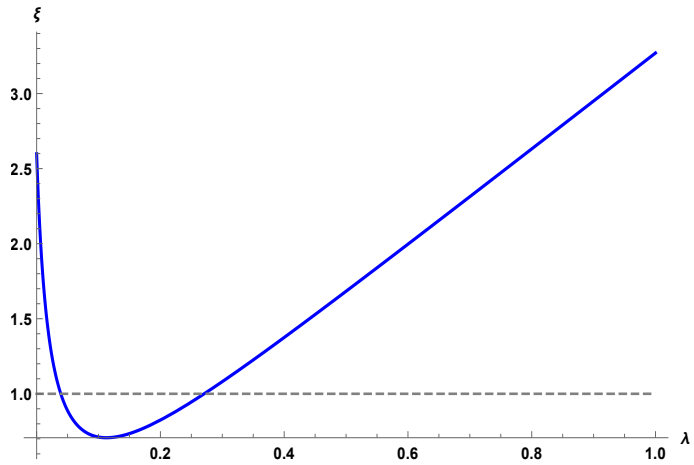

Figure 2: The ratio ξ when $c = 0.4$, $\gamma = 3$, $\bar{\sigma}^2 = 0.1$, $\underline{\sigma}^2 = 0.04$, $\sigma_\varepsilon^2 = 0.07$, $\sigma_z^2 = 0.05$. We find two information market equilibria at $\lambda = 0.0390$ and $\lambda = 0.2707$. The former is the equilibrium with strategic complementarity and the latter is that with strategic substitutability.

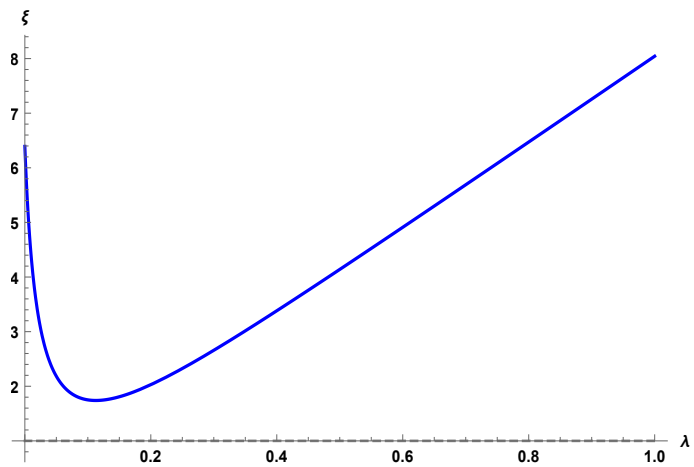

Figure 3: The ratio ξ when $c = 0.7$, $\gamma = 3$, $\bar{\sigma}^2 = 0.1$, $\underline{\sigma}^2 = 0.04$, $\sigma_\varepsilon^2 = 0.07$, $\sigma_z^2 = 0.05$. Since $\xi > 0$ for $\lambda \in [0, 1]$, in equilibrium, all traders become uninformed, in which there is neither strategic complementarity nor strategic substitutability.

4.2 Overall Equilibrium

Now, by 3.1 amd 4.4, we derive overall equilibrium backwardly.

Proposition 4.5. *The overall equilibrium is given as the following.*

1. *If $c \leq \underline{c}$, then $\lambda^* = 1$ and the equilibrium price is*
$$P^* = \theta - \gamma\sigma_\varepsilon^2 z.$$
2. *If $c \in (\underline{c}, \bar{c})$, there exists a unique equilibrium $\lambda^* \in (0,1)$ and the equilibrium price is*
$$P^* = (1-\alpha^*)\mu + \alpha^* s^*$$
where
$$\alpha^* = \frac{\lambda^*(\lambda^*\bar{\sigma}^2 + \gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2)}{(\lambda^*)^2\bar{\sigma}^2 + \lambda^*\gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2}.$$
3. *If $c = \bar{c}$, there exist two equilibria: $\lambda_1^* = 0$ and $\lambda_2^* \in (0,1)$. At λ_1^*, the equilibrium price is*
$$P^* = \mu - \gamma(\bar{\sigma}^2 + \sigma_\varepsilon^2)z$$
and at λ_2^*, the equilibrium price is
$$P^* = (1-\alpha^*)\mu + \alpha^* s^*$$
where
$$\alpha^* = \frac{\lambda_2^*(\lambda_2^*\bar{\sigma}^2 + \gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2)}{(\lambda_2^*)^2\bar{\sigma}^2 + \lambda_2^*\gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2}.$$
4. *If $c \in (\bar{c}, \hat{c})$, there exist two equilibria: $\lambda_1^* \in (0,1)$ and $\lambda_2^* \in (0,1)$ where $\lambda_1^* < \lambda_2^*$. At λ_1^*, the equilibrium price is*
$$P^* = (1-\alpha_1^*)\mu + \alpha_1^* s^*$$
where
$$\alpha_1^* = \frac{\lambda_1^*(\lambda_1^*\bar{\sigma}^2 + \gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2)}{(\lambda_1^*)^2\bar{\sigma}^2 + \lambda_1^*\gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2}.$$
and at λ_2^*, the equilibrium price is
$$P^* = (1-\alpha_2^*)\mu + \alpha_2^* s^*$$
where
$$\alpha^* = \frac{\lambda_2^*(\lambda_2^*\bar{\sigma}^2 + \gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2)}{(\lambda_2^*)^2\bar{\sigma}^2 + \lambda_2^*\gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2}.$$
5. *If $c = \hat{c}$, there exist a unique equilibrium $\lambda^* = \hat{\lambda}$ and the equilibrium price is given by*
$$P^* = (1-\hat{\alpha}^*)\mu + \hat{\alpha}^* s^*$$
where
$$\alpha^* = \frac{\hat{\lambda}^*(\hat{\lambda}^*\bar{\sigma}^2 + \gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2)}{\hat{\lambda}^2\bar{\sigma}^2 + \hat{\lambda}\gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2}.$$
6. *If $c > \hat{c}$, then there exists a unique equilibrium $\lambda^* = 0$ and the equilibrium price is*
$$P^* = \mu - \gamma(\bar{\sigma}^2 + \sigma_\varepsilon^2)z$$

5 Conclusion

In this paper, we analyze the model with endogenous information acquisition when ex ante identical traders initially have ambiguous information about the variance. Facing ambiguity, would-be informed traders and would-be uninformed traders consider different probability distributions when they maximize their ex ante expected utility. As the worst case scenarios, would-be informed traders consider the minimum variance, while would-be uninformed traders consider the maximum variance. This difference leads to the possibility of multiple equilibria. We find that if the cost of information lies in some range, both equilibria with strategic substitutability and with strategic complementarity coexist in information markets. The increase of information cost decreases the proportion informed traders in equilibrium with strategic substitutability and increases it in equilibrium with strategic complementarity.

The effects of ambiguity on information markets are also examined. In the equilibria with strategic substitutability, as the minimum variance of the true value decreases, the proportion of informed traders decreases, while as the maximum variance of the true increases, the proportion of informed traders decreases. In equilibria with strategic complementarity, changes of the minimum variance and the maximum variance reversely affect the proportion of informed traders.

Appendix

Proof of Proposition 3.1: First, we suppose $\lambda = 0$. Then the market clearing condition (3.1) reduces to $x_u = z$ and compound signal $\tilde{s}$ is given by $\tilde{s}(\theta, z) = -z$. We conjecture that the equilibrium price P_0 when $\lambda = 0$ is a linear function of s:

$$P_0(s) = \mu + \alpha_0 s, \quad \forall\, s \in \mathbb{R}.$$

Then information generated by the equilibrium price $\tilde{p}$ is equivalent to that from $\tilde{s}$. Let σ_π^2 is the variance of θ under probability distribution $\pi \in \mathcal{P}$. For every $\pi \in \mathcal{P}$, we have

$$\begin{aligned}\mathbb{E}_\pi[\tilde{v}|\tilde{p} = p] &= \mathbb{E}_\pi[\tilde{v}|\tilde{s} = s] = \mathbb{E}[v] = \mu,\\ \text{Var}_\pi[\tilde{v}|\tilde{p} = p] &= \text{Var}_\pi[\tilde{v}|\tilde{s} = s] = \text{Var}[v] = \sigma_\pi^2 + \sigma_\varepsilon^2.\end{aligned}$$

Note that $\mathbb{E}_\pi[\tilde{v}|\tilde{s} = s]$ is independent of σ_π^2 and $\text{Var}_\pi[\tilde{v}|\tilde{p} = p]$ increases in σ_π^2. From (2.2), the uninformed traders' problem becomes

$$\max_{\hat{x}_u} \min_{\pi \in \mathcal{P}} \left[\gamma \left(w_0 + \hat{x}_u(\mu - p) - \frac{1}{2}\gamma \hat{x}_u^2(\sigma_\pi^2 + \sigma_\varepsilon^2)\right)\right].$$

Allowing for the worse case probability distribution, the uninformed traders set $\sigma_\pi^2 = \bar{\sigma}^2$, and their demand function becomes

$$x_u(\tilde{p}) = \frac{\mu - p}{\gamma(\bar{\sigma}^2 + \sigma_\varepsilon^2)}, \quad p \in \mathbb{R}.$$

Now the market clearing condition (3.1) can be written as

$$\frac{\mu - p}{\gamma(\bar{\sigma}^2 + \sigma_\varepsilon^2)} = z,$$

which yields

$$P_0(s) = \mu + \alpha_0 s$$

where $a_0 = \gamma(\bar{\sigma}^2 + \sigma_\varepsilon^2)$.

Now we suppose $\lambda \in (0, 1]$. Then compound signal $\tilde{s}$ is given by

$$\tilde{s}(\theta, z) = \theta - \frac{\gamma\sigma_\varepsilon^2}{\lambda} z.$$

We conjecture that P is a linear function of s such that

$$P_\lambda(s) = (1 - \alpha_\lambda)\mu + \alpha_\lambda s$$

where

$$\alpha_\lambda = \frac{\lambda(\lambda\sigma_\pi^2 + \gamma^2\sigma_\pi^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2)}{\lambda^2\sigma_\pi^2 + \lambda\gamma^2\sigma_\pi^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2}.$$

Then information generated by the equilibrium price $\tilde{p}$ is equivalent to that by $\tilde{s}$. Since $\tilde{s}$ and $\tilde{v}$ are normally distributed, for every $\pi \in \mathcal{P}$, we have

$$\begin{aligned} \mathbb{E}_\pi[\tilde{v}|\tilde{p} = p] = \mathbb{E}_\pi[\tilde{v}|\tilde{s} = s] &= \frac{\gamma^2\sigma_\varepsilon^4\sigma_z^2\mu + \lambda^2\sigma_\pi^2 s}{\lambda^2\sigma_\pi^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2} = (1 - \beta_\pi)\mu + \beta_\pi s, \\ \mathrm{Var}_\pi[\tilde{v}|\tilde{p} = p] = \mathrm{Var}_\pi[\tilde{v}|\tilde{s} = s] &= \frac{\sigma_\varepsilon^2(\lambda^2\sigma_\pi^2 + \gamma^2\sigma_\pi^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2)}{\lambda^2\sigma_\pi^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2} \end{aligned} \tag{A.1}$$

where

$$\beta_\pi \equiv \frac{\lambda^2\sigma_\pi^2}{\lambda^2\sigma_\pi^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2}.$$

Since

$$\begin{aligned} \frac{\partial\mathbb{E}_\pi[\tilde{v}|\tilde{s} = s]}{\partial\sigma_\pi^2} &= \frac{\lambda^2\gamma^2\sigma_\varepsilon^4\sigma_z^2(s - \mu)}{(\lambda^2\sigma_\pi^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2)^2}, \\ \frac{\partial\mathrm{Var}_\pi[\tilde{v}|\tilde{s} = s]}{\partial\sigma_\pi^2} &= \frac{\gamma^4\sigma_\varepsilon^8\sigma_z^4}{(\lambda^2\sigma_\pi^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2)^2} > 0, \end{aligned} \tag{A.2}$$

we know that $\mathbb{E}_\pi[\tilde{v}|\tilde{s} = s]$ decreases (increase) in σ_π^2 if $s < \mu$ ($s > \mu$, respectively) and $\mathrm{Var}[\tilde{v}|\tilde{s} = s]$ always increases in $\bar{\sigma}^2$ for every $\pi \in \mathcal{P}$. Then we have Lemma A.1.

Lemma A.1. *For every $\pi \in \mathcal{P}$, the following hold.*

1. *The conditional expectation $\mathbb{E}_\pi[\tilde{v}|\tilde{s} = s]$ strictly increases (decreases) in σ_π^2 if and only if $s > \mu$ ($s < \mu$, respectively).*

2. *The conditional variance $\mathrm{Var}_\pi[\tilde{v}|\tilde{s} = s]$ increases in σ_π^2.*

The difference between the conditional expectation $\mathbb{E}_\pi[\tilde{v}|\tilde{s}=s]$ and asset price p is given by

$$\mathbb{E}_\pi[\tilde{v}|\tilde{s}=s]-p=(1-\beta_\pi)+\beta_\pi s-[(1-\alpha_\lambda)\mu+\alpha_\lambda s]=(\alpha_\lambda-\beta_\pi)(\mu-s). \tag{A.3}$$

Since

$$\alpha_\lambda-\beta_\pi=\frac{\lambda\gamma^2\sigma_\varepsilon^4\sigma_z^2(\lambda^2\sigma_\pi^2+\gamma^2\sigma_\pi^2\sigma_\varepsilon^2\sigma_z^2+\gamma^2\sigma_\varepsilon^4\sigma_z^2)}{(\lambda^2\sigma_\pi^2+\gamma^2\sigma_\varepsilon^4\sigma_z^2)(\lambda^2\sigma_\pi^2+\lambda\gamma^2\sigma_\pi^2\sigma_\varepsilon^2\sigma_z^2+\gamma^2\sigma_\varepsilon^4\sigma_z^2)}>0,$$

the difference $\mathbb{E}_\pi[\tilde{v}|\tilde{s}=s]-p$ is higher than zero if and only if $s<\mu$, which implies Lemma A.2.

Lemma A.2. *For every $\pi\in\mathcal{P}$, the following hold.*

1. *The inequality $p<\mathbb{E}_\pi[\tilde{v}|\tilde{s}=s]$ ($p>\mathbb{E}_\pi[\tilde{v}|\tilde{s}=s]$) is equivalent to $s<\mu$ ($s>\mu$, respectively).*
2. *The equality $p=\mathbb{E}_\pi[\tilde{v}|\tilde{s}=s]$ is equivalent to $s=\mu$.*

Recall (2.3) and we consider three cases: (i) $p<\mathbb{E}_{\hat{\pi}}[\tilde{v}|\tilde{p}=p]$, (ii) $p>\mathbb{E}_{\bar{\pi}}[\tilde{v}|\tilde{p}=p]$, and (iii) $\mathbb{E}_{\hat{\pi}}[\tilde{v}|\tilde{p}=p]\le p\le\mathbb{E}_{\bar{\pi}}[\tilde{v}|\tilde{p}=p]$.

(i) Note that $p<\mathbb{E}_{\hat{\pi}}[\tilde{v}|\tilde{p}=p]$ is equivalent to $s<\mu$ by the first claim of Lemma A.2. Thus, probability distribution $\hat{\pi}\in\mathcal{P}$ which minimizes $\mathbb{E}_\pi[\tilde{v}|\tilde{p}=p]$ and maximizes $\mathrm{Var}_\pi[\tilde{v}|\tilde{p}=p]$ takes the maximum variance $\bar{\sigma}^2$ by Lemma A.1. Plugging $\bar{\sigma}^2$ into the uninformed traders' demand function (2.3), we solve the market clearing condition (3.1) for p to obtain the equilibrium asset price, given by

$$\begin{aligned}\hat{P}_\lambda(s)&=\frac{(1-\lambda)\sigma_\varepsilon^2\mathbb{E}_{\hat{\pi}}[\tilde{v}|\tilde{p}=p]+\lambda s\mathrm{Var}_{\hat{\pi}}[\tilde{v}|\tilde{p}=p]}{(1-\lambda)\sigma_\varepsilon^2+\lambda\mathrm{Var}_{\hat{\pi}}[\tilde{v}|\tilde{p}=p]}\\&=\frac{(1-\lambda)\gamma^2\sigma_\varepsilon^4\sigma_z^2}{\lambda^2\bar{\sigma}^2+\lambda\gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2+\gamma^2\sigma_\varepsilon^4\sigma_z^2}\mu+\frac{\lambda(\lambda\bar{\sigma}^2+\gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2+\gamma^2\sigma_\varepsilon^4\sigma_z^2)}{\lambda^2\bar{\sigma}^2+\lambda\gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2+\gamma^2\sigma_\varepsilon^4\sigma_z^2}s\\&=(1-\alpha)\mu+\alpha s\end{aligned} \tag{A.4}$$

for $s<\mu$.

(ii) Note that $p>\mathbb{E}_{\bar{\pi}}[\tilde{v}|\tilde{p}=p]$ is equivalent to $s>\mu$ by the first claim of Lemma A.2. Thus, probability distribution $\bar{\pi}\in\mathcal{P}$ which maximizes both $\mathbb{E}_\pi[\tilde{v}|\tilde{p}=p]$ and $\mathrm{Var}_\pi[\tilde{v}|\tilde{p}=p]$ takes the maximum variance $\bar{\sigma}^2$ by Lemma A.1. Similar to case (i), we obtain the equilibrium asset price, given by

$$\bar{P}_\lambda(s)=(1-\alpha)\mu+\alpha s\quad\text{for}\quad s>\mu.$$

(iii) From cases (i) and (ii), we know $\mathbb{E}_{\bar{\pi}}[\tilde{v}|\tilde{s}=p]=\mathbb{E}_{\hat{\pi}}[\tilde{v}|\tilde{p}=p]$ and thus $p\in[\mathbb{E}_{\bar{\pi}}[\tilde{v}|\tilde{s}=p],\mathbb{E}_{\hat{\pi}}[\tilde{v}|\tilde{p}=p]]$ is equivalent $s=\mu$ by the second claim of Lemma A.2. Solving (3.1) for p, we have the equilibrium asset price, given by

$$P'_\lambda(s)=\mu\quad\text{for}\quad s=\mu.$$

From cases (i), (ii) and (iii), for $\lambda \in (0,1]$, we have equilibrium asset price function

$$P_\lambda(s) = (1-\alpha)\mu + \alpha s, \quad \forall\, s \in \mathbb{R}$$

where

$$\alpha = \frac{\lambda(\lambda\bar{\sigma}^2 + \gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2)}{\lambda^2\bar{\sigma}^2 + \lambda\gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2}.$$

Furthermore, since

$$\lim_{\lambda\to 0^+} P_\lambda(s) = P_0(s),$$

for $\lambda \in [0,1]$, we express the the equilibrium asset price function as

$$P(s) = (1-\alpha)\mu + \alpha s, \quad \forall\, s \in \mathbb{R}. \qquad \blacksquare$$

Proof of Proposition 4.1: (1) Let $\pi_i \in \mathcal{P}$ be the probability distribution informed trader i considers at the ex ante stage. Then i's ex ante expected utility is given by

$$U_i(w_i) = \min_{\pi_i \in \mathcal{P}} \mathbb{E}_{\pi_i}[-e^{-\gamma w_i}].$$

Interim expected utility of would-be informed trader i is given by

$$\begin{aligned}
\mathbb{E}_{\pi_i}[u(w_i)|p,\theta] &= -\exp\left[-\gamma\left(w_0 - c + x_i(\mathbb{E}_{\pi_i}[\tilde{v}|p,\theta] - p) - \frac{\gamma}{2}x_i^2\mathrm{Var}_{\pi_i}[\tilde{v}|p,\theta]\right)\right] \\
&= -\exp\left[-\gamma\left(w_0 - c + x_i(\theta - p) - \frac{\gamma}{2}x_i^2\sigma_\varepsilon^2\right)\right] \\
&= e^{\gamma c}u(w_0)\left[-\gamma\left(x_i(\theta - p) - \frac{\gamma}{2}x_i^2\sigma_\varepsilon^2\right)\right].
\end{aligned}$$

By (2.1), we obtain

$$\begin{aligned}
\mathbb{E}_{\pi_i}[u(w_i)|p,\theta] &= e^{\gamma c}u(w_0)\exp\left[-\gamma\left(\frac{(\theta-p)^2}{\gamma\sigma_\varepsilon^2} - \frac{(\theta-p)^2}{2\gamma\sigma_\varepsilon^2}\right)\right] \\
&= e^{\gamma c}u(w_0)\exp\left[-\frac{(\theta-p)^2}{2\sigma_\varepsilon^2}\right].
\end{aligned}$$

Since $\mathbb{E}_{\pi_i}[\mathbb{E}_{\pi_i}[u_i(w_i)|p,\theta]|s] = \mathbb{E}_{\pi_i}[u_i(w_i)|s]$, we have

$$\begin{aligned}
\mathbb{E}_{\pi_i}[u_i(w)|s] &= e^{\gamma c}u(w_0)\mathbb{E}_{\pi_i}\left[\exp\left(-\frac{(\mathbb{E}_{\pi_i}[\tilde{v}|p,\theta] - P(s))^2}{2\sigma_\varepsilon^2}\right)\middle| s\right] \\
&= e^{\gamma c}u(w_0)\sqrt{\frac{\sigma_\varepsilon^2}{\mathrm{Var}_{\pi_i}[\tilde{v}|s]}}\exp\left(-\frac{(\mathbb{E}_{\pi_i}[\tilde{v}|s] - P(s))^2}{2\mathrm{Var}_{\pi_i}[\tilde{v}|s]}\right)
\end{aligned}$$

By (A.3),

$$\mathbb{E}_{\pi_i}[u_i(w)|s] = e^{\gamma c}u(w_0)\sqrt{\xi_i}\exp\left(-\frac{(\alpha - \beta_{\pi_i})^2}{2\mathrm{Var}_{\pi_i}[\tilde{v}|s]}(\mu - s)^2\right)$$

where

$$\xi_{\pi_i} \equiv \frac{\lambda^2\sigma_{\pi_i}^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2}{\lambda^2\sigma_{\pi_i}^2 + \gamma^2\sigma_{\pi_i}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2},$$
$$\beta_{\pi_i} \equiv \frac{\lambda^2\sigma_{\pi_i}^2}{\lambda^2\sigma_{\pi_i}^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2}.$$

Let $(\sigma_{\pi_i}^s)^2$ denote the variance of compound signal s under probability distribution π_i. We define

$$f_{\pi_i}(s) \equiv \frac{1}{\sqrt{2\pi}\sigma_{\pi_i}^s}\exp\left(-\frac{(s-\mu)^2}{2(\sigma_{\pi_i}^s)^2}\right).$$

For a given $\sigma_{\pi_i}^2 \in [\underline{\sigma}^2, \bar{\sigma}^2]$, the would-be informed trader i' ex ante expected utility is

$$\begin{aligned}
\mathbb{E}[u(w_i)] &= e^{\gamma c}u(w_0)\sqrt{\xi_{\pi_i}}\int_{-\infty}^{\infty}\exp\left(-\frac{(\alpha-\beta_{\pi_i})^2}{2\text{Var}_{\pi_i}[\tilde{v}|s]}(s-\mu)^2\right)f_{\pi_i}(s)ds \\
&= e^{\gamma c}u(w_0)\frac{\sqrt{\xi_{\pi_i}}}{\sqrt{2\pi}\sigma_{\pi_i}^s}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{(\alpha-\beta_{\pi_i})^2}{2\text{Var}_{\pi_i}[\tilde{v}|s]}+\frac{1}{2(\sigma_{\pi_i}^s)^2}\right)(s-\mu)^2\right]ds \\
&= e^{\gamma c}u(w_0)\frac{\sqrt{\xi_{\pi_i}}}{\sqrt{2\pi}\sigma_{\pi_i}^s}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{(\alpha-\beta_{\pi_i})^2(\sigma_{\pi_i}^s)^2+\text{Var}_{\pi_i}[\tilde{v}|s]}{2(\sigma_{\pi_i}^s)^2\text{Var}_{\pi_i}[\tilde{v}|s]}\right)(s-\mu)^2\right]ds \\
&= e^{\gamma c}u(w_0)\sqrt{\frac{\sigma_\varepsilon^2}{(\alpha-\beta_{\pi_i})^2(\sigma_{\pi_i}^s)^2+\text{Var}_{\pi_i}[\tilde{v}|s]}} \\
&= e^{\gamma c}u(w_0)\sqrt{h_{\pi_i}}
\end{aligned}$$

where

$$h_{\pi_i} \equiv \frac{(\lambda^2\bar{\sigma}^2 + \lambda\gamma^2\bar{\sigma}^2\sigma_\varepsilon^2\sigma_z^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2)^2}{X_i}$$

with

$$X_i \equiv \left[(1-\lambda)^2\sigma_{\pi_i}^2 + \gamma^2\sigma_\varepsilon^4\sigma_z^2 + \sigma_\varepsilon^2\right]\gamma^4\sigma_\varepsilon^6\sigma_z^4 + (\lambda+\gamma^2\sigma_\varepsilon^2\sigma_z^2)^2\left[(\lambda^2+\gamma^2\sigma_\varepsilon^2\sigma_z^2)\bar{\sigma}^2 + 2\gamma^2\sigma_\varepsilon^4\sigma_z^2\right]\bar{\sigma}^2.$$

Since h_{π_i} decreases in $\sigma_{\pi_i}^2$ and $u(w_0) < 0$, the would-be informed traders' ex ante expected utility is minimized at $\sigma_{\pi_i}^2 = \underline{\sigma}^2$ and is given by

$$U_i(\lambda) = e^{\gamma c}u(w_0)\sqrt{h_i}. \qquad \blacksquare$$

(2) Let $\pi_u \in \mathcal{P}$ be the probability distribution would-be uninformed trader u considers at the ex ante stage. Then u's ex ante expected utility is given by

$$U_u(w_u) = \min_{\pi_u\in\mathcal{P}}\mathbb{E}_{\pi_u}[-e^{-\gamma w_u}].$$

Interim expected utility of would-be uninformed trader u is given by

$$\begin{aligned}
\mathbb{E}_{\pi_u}[u(w_u)|\tilde{s}] &= -\exp\left[-\gamma\left(w_0 + x_u(\mathbb{E}_{\pi_u}[\tilde{v}|\tilde{s}]-p) - \frac{\gamma}{2}x_u^2\text{Var}_{\pi_u}[\tilde{v}|\tilde{s}]\right)\right] \\
&= u(w_0)\mathbb{E}_{\pi_u}\left[\exp\left(-\xi_u(\mu-s)^2\right)\right]
\end{aligned}$$

where

$$\xi_u \equiv \gamma\kappa\left(\alpha - \beta_{\pi_u} - \frac{\gamma\kappa}{2}\text{Var}_{\pi_u}[\tilde{v}|\tilde{s}]\right).$$

Let us define

$$f_{\pi_u}(s) \equiv \frac{1}{\sqrt{2\pi}\sigma^s_{\pi_u}} \exp\left(-\frac{(s-\mu)^2}{2(\sigma^s_{\pi_u})^2}\right)$$

where

$$(\sigma^s_{\pi_u})^2 = \sigma^2_{\pi_u} + \frac{\gamma^2\sigma^4_\varepsilon\sigma^2_z}{\lambda^2}.$$

For a given $\sigma^2_{\pi_u} \in [\underline{\sigma}^2, \bar{\sigma}^2]$, we have

$$\begin{aligned}
\mathbb{E}[u(w_u)] &= u(w_0)\int_{-\infty}^{\infty} \exp\left(-\xi_u(\mu-s)^2\right) f_{\pi_u}(s)ds \\
&= \frac{u(w_0)}{\sqrt{2\pi}\sigma^s_{\pi_u}}\int_{-\infty}^{\infty} \exp\left[-\frac{1+2\xi_u(\sigma^s_{\pi_u})^2}{2(\sigma^s_{\pi_u})^2}(\mu-s)^2\right] ds \\
&= \frac{u(w_0)}{\sqrt{1+2_u(\sigma^s_{\pi_u})^2}} \\
&= u(w_0)\sqrt{h_{\pi_u}}
\end{aligned}$$

where

$$h_{\pi_u} \equiv \frac{(\lambda\bar{\sigma}^2 + \lambda\gamma^2\bar{\sigma}^2\sigma^2_\varepsilon\sigma^2_z + \gamma^2\sigma^4_\varepsilon\sigma^2_z)^2}{X_u}$$

with

$$X_u \equiv \lambda^4\bar{\sigma}^4 + \gamma^2\sigma^2_\varepsilon\sigma^2_z[2\lambda^2\bar{\sigma}^2(\lambda\bar{\sigma}^2+\sigma^2_\varepsilon) + \gamma^2\sigma^2_\varepsilon\sigma^2_z[\lambda^2\bar{\sigma}^4 + \sigma^2_\varepsilon\{4\lambda\bar{\sigma}^2 - (2-\lambda)\lambda\sigma^2_{\pi_u} + \gamma^2\sigma^2_\varepsilon\sigma^2_z(2\bar{\sigma}^2 - \sigma^2_{\pi_u} + \sigma^2_\varepsilon) + \sigma^2_\varepsilon\}]].$$

Since h_{π_u} increases in $\sigma^2_{\pi_u}$ and $u(w_0) < 0$, the would-be uninformed traders' ex ante expected utility is minimized at $\sigma^2_{\pi_u} = \bar{\sigma}^2$ and is given by

$$U_u(\lambda) = e^{\gamma c}u(w_0)\sqrt{h_u}. \qquad \blacksquare$$

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