Central Bank Digital Currency, Inflation Tax, and Central Bank Independence*

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Abstract

Can introducing Central Bank Digital Currency (CBDC) improve social welfare? We construct a dual currency model to study whether introducing CBDC with a record-keeping technology can reduce tax evasion incentives in cash transactions, and further achieve a better allocation than in a cash-only economy. Tax evasion does not occur in an economy only with an inflation tax. However, if imposing a positive sales tax is inevitable for central bank independence, there arises an inefficiency associated with tax evasion in cash transactions. Introducing CBDC with positive interest can reduce this inefficiency and thus improve welfare by discouraging tax evasion, and rewarding tax payments.

Keywords: cash, central bank digital currency, monetary policy, inflation tax, tax evasion

JEL classification: E31, E42, E58, H21, H26

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1 Introduction

Recently, Central Bank Digital Currency (henceforth, CBDC) has inspired extensive research among central banks.[1] CBDC differs from cash, which is issued in physical paper or coin form, in the sense that all transaction information can be recorded in a digital ledger kept by the central bank. Obviously, this digital ledger of CBDC can be shared with fiscal authorities for tax collection. Moreover, CBDC can bear interest because it is possible to verify its ownership using the digital ledger.

Because of its features, one may argue that the government should introduce CBDC in order to reduce tax evasion—which can easily occur in cash transactions—and thus increase sales tax revenues. On the other hand, one can claim that CBDC is not necessary to reduce tax evasion, because the central bank can levy an inflation tax to cash users instead of sales tax. Historically, it is true that central bank seigniorage has been used for government spending when its tax revenue was not enough to finance expenditures: for example, when tax evasion is severe or a war is underway. In this respect, the effects of introducing CBDC could be closely related to central bank independence (henceforth, CBI) from the fiscal authority. According to Sargent (1982), a transfer of seigniorage from the central bank to the fiscal authority affects public expectations about inflation and thus real allocations.[2] However, our understanding is still limited regarding how the introduction of CBDC contributes to an economy in which tax evasion occurs and CBI matters.

This study aims to examine how the introduction of CBDC can affect welfare in an economy in which cash is used as a medium of exchange (henceforth, MOE) and tax evasion exists in cash transactions. As long as there are substantial tax evasion-related benefits from the use of cash, CBDC might not completely replace cash. However, introducing CBDC with a positive interest rate can encourage record-keeping trades and reduce the incentive

[1] For example, the Bank of England, the Bank of Canada, the People's Bank of China, and the Sveriges Riksbank have been actively exploring the possibility of issuing CBDC. See Barontini and Holden (2019) for survey results of sixty three central banks' stance toward CBDC.

[2] In this paper, CBI is referred to as a case in which no monetary transfers are allowed from the central bank to the fiscal authority. We lean toward Sargent (1982) rather than Williamson (2019), who articulates another aspect of CBI. Sargent (1982) raises concerns over transferring seigniorage from the central bank to the fiscal authority. On the other hand, Williamson (2019) is concerned about the case in which interest payments on CBDC could threaten CBI since the central bank's budget is likely to rely on support from the government.

to evade sales tax. Exploring how CBDC can improve welfare, along with CBI, will provide more knowledge about the impact of a record-keeping MOE on tax evasion.

We build up a monetary model, based on Lagos and Wright (2005) and Williamson (2012), in which cash and CBDC can be used as MOE in pairwise meetings, and MOE choices for transactions are endogenously determined by the relative rates of return on the two monies. A fraction of meetings are monitored by the fiscal and monetary authority, whereas the remainder of the meetings is not monitored. In the monitored meetings, the fiscal authority can levy a proportional sales tax on transactions regardless of MOE type. In the non-monitored meetings, it specifically depends on MOE type. If CBDC is used as a MOE, the fiscal authority can impose tax, but if cash is used as a MOE, then the authority cannot impose tax, so tax evasion can occur. There are two ways that the fiscal authority finances expenditures: collecting sales tax and/or receiving seigniorage revenue from the central bank. Finally, the central bank can adjust the cash supply and CBDC and thus the relative rates of return on the two monies.

In this environment, we first study the equilibrium conditions for the coexistence of cash and CBDC and then equilibrium types, depending on the rates of return on cash and CBDC: a pooling equilibrium where only one type of money is used in both meetings, a partially pooling equilibrium where both monies are used together in either non-monitored or monitored meetings, a separating equilibrium where cash is used only in non-monitored meetings and CBDC is used only in monitored meetings. For instance, if the rate of return on cash is higher than the rate of return on CBDC allowing for sales tax, then each money is used separately: cash, which makes it possible to evade taxation, will be used in non-monitored meetings and CBDC will be used in monitored meetings.

More importantly, we present that whether CBDC is beneficial for welfare depends on the CBI. We compare welfare implications of two economies, a cash-only economy and an economy in which cash and CBDC coexist by addressing the optimal monetary and fiscal policy mix. First, in an economy without CBI, introducing CBDC does not improve welfare. When cash is the only MOE in the economy and tax evasion occurs, welfare can be maximized by substituting a sales tax for an inflation tax. Specifically, the fiscal authority can lower the sales tax rate to zero and receive the seigniorage revenue from the central bank to finance its expenditures. This policy mix can eliminate sales tax distortion and achieve the efficient allocation. In this case, even though CBDC is introduced, the zero percent sales tax rate is still optimal, and both cash and CBDC will be used in a partially pooling equilibrium. Introducing CBDC does not change the optimal equilibrium allocation.

On the other hand, in an economy with CBI, introducing CBDC can improve welfare. Since a seigniorage transfer is not available, it is necessary for the fiscal authority to impose a proportional sales tax, which inevitably causes two types of tax distortion. One is a distortion in the relative marginal utility between non-monitored and monitored meetings. The other is a loss of tax revenue, because sales tax is less efficient than inflation tax for financing a certain level of government spending. The central bank can correct for the former type of distortion by implementing a higher growth rate of cash than that of CBDC and/or paying a strictly positive nominal interest on CBDC. By collecting more inflation tax from cash transactions and less from CBDC transactions, the central bank can make an implicit transfer from those who evade taxes in non-monitored meetings to those who pay taxes in monitored. However, the loss in the tax revenue cannot be perfectly restored—although the sales tax rate can decrease—because the fixed amount of government expenditures must only be supported by the sales tax. In sum, CBI can rationalize the introduction of CBDC with a positive nominal interest rate in the sense that it is beneficial for welfare.

1.1 Related Literature

This paper is closely related to previous studies such as Williamson (2019) and Davoodalhosseini (2018). Williamson (2019) develops a model of multiple means of payments to examine the implications of introducing CBDC. He incorporates a crime associated with cash such as theft and shows that introducing CBDC can have benefits: CBDC can mitigate cash-associated crime and also economize the scarcity of safe collateral. Furthermore, he raises the issue that paying interest on CBDC can pose a threat to CBI. In a similar model with a fixed CBDC acquisition cost, Davoodalhosseini (2018) provides a condition under which introducing CBDC can implement the first-best allocation. We complement their work by introducing a proportional sales tax for incentivizing tax evasion, and adopting the notion of CBI explicitly. Specifically, we consider the proportional sale tax as a cost explicitly associated with CBDC instead of a implicit fixed cost of CBDC as shown in Davoodalhosseini (2018). In our paper, CBDC plays a role as an MOE that discourages tax evasion. Also, we show that CBI is one of the crucial elements to understand the welfare effect of CBDC.

There also has been a growing body of literature on CBDC.[3] Sanches and Keister (2019) construct a model in which an interest-bearing CBDC plays the role of an efficient MOE and show that the introduction of CBDC can increase welfare despite its negative impact on financial intermediation. Andolfatto (2018) and Chiu et al. (2019) develop models in which an interest-bearing CBDC competes with commercial bank deposits and show that the introduction of CBDC increases welfare if the banks' market power in the deposit market is limited by CBDC. Meanwhile, Brunnermeier and Niepelt (2019) and Kim and Kwon (2019) construct models in which commercial banks provide liquidity to examine the impact of CBDC on financial stability. They claim that if the introduction of CBDC and central bank pass-through funding go hand in hand, CBDC need not result in a credit crunch or undermine financial stability.

This paper also relates to the work on tax evasion and optimal inflation. Gomis-Porqueras et al. (2014) provides a theory to guide the measurement of the underground sector of economy. Gomis-Porqueras et al. (2014) focus on measuring the size of the underground sector, whereas we emphasize an optimal tax scheme under tax evasion. In a similar vein, Koreshkova (2006) quantitatively examines the public finance motive for inflation to explain high inflation rates in developing countries. The paper finds a negative relationship between inflation and the size of the underground economy, and shows that the government may optimally choose a high inflation rate when the underground sector exists. Nicolini (1998) also studies how tax evasion affects the optimal inflation tax and shows that the optimal interest rate is positive when cash is used for underground transactions. We consider an optimal fiscal and monetary policy mix and examine it in an economy with two types of money: cash and CBDC. The transfer from cash users to CBDC users through seigniorage revenue can correct the distortion caused by the proportional sales tax and tax evasion in

[3] For discussions on CBDC including its motivations and implications, see Barrdear and Kumhoff (2018), Bech and Garratt (2017), Bordo and Levin (2017), Broadbent (2016), Dyson and Hodgson (2017), Engert and Fung (2017), Fung and Halaburda (2017), Raskin and Yermack (2016), and Ricks et al. (2018).

an economy in which CBI is required.

Our model approach is related to recent papers on seigniorage with dual currencies. Zhang (2014) studies international currency competition by considering the role of seigniorage. Hendrickson and Park (2018) show that dual currencies can be useful to improve welfare by making an implicit transfer from one type of MOE user to another. We extend their model to analyze the effect of seigniorage transfers but focus on the role of record-keeping money on tax evasion incentives along with CBI.

Finally, our work is also related to literature on CBI.[4] There are many empirical studies that show a negative relationship between the average inflation rate and measures for CBI.[5] For example, Alesina and Summers (1993) find out that advanced countries with high levels of CBI experienced lower levels of inflation during the period between 1955 and 1988. On the other hand, a few papers address CBI in a theoretical manner. Rogoff (1985) shows that introducing the central bank, which puts more weight on inflation objectives, can improve outcomes. Instead of dealing with the inflation bias, in this paper, we emphasize the transfer between the central bank and the fiscal authority to capture CBI. With respect to fiscal support, Sargent and Wallace (1981) is the seminal work that examines how a fiscal authority can force a central bank to generate more seigniorage when it fails to finance the budget deficit. Recently, Martin (2015) shows that CBI might temporarily, but not permanently, decrease the inflation rate in a dynamic model in which the fiscal authority controls taxes and expenditures, whereas the central bank manages monetary policy separately. In this paper, we also consider the optimal fiscal and monetary policy mix with fiscal budget constraints, but we compare equilibrium outcomes in the long run rather than investigate a trade-off in the dynamics.

The remainder of this paper is organized as follows. Section 2 describes the model economy and Section 3 analyzes the equilibrium conditions in which cash and CBDC can co-exist. Section 4 investigates the welfare implications of CBDC. Section 5 concludes with a few remarks.

[4] See Walsh (2008, 2011), and Waller (2011) to understand the various concepts on CBI. See Haan and Eijffinger (2016) for a survey of the literature.

[5] See Cukierman (1992) for the survey on the empirical work.

2 The Model

The basic structure is built on Lagos and Wright (2005) and Williamson (2012), in which quasi-linear preferences make a model analytically tractable with an array of assets. Time is discrete and goes on forever, indexed by $t = 1, 2, ...$ In each time period, a centralized and competitive market (henceforth, CM) and a decentralized market (henceforth, DM) open sequentially. There is a discount factor, $\beta \in (0, 1)$, between periods. There are two types of continua of agents, buyers and sellers, each with unit mass. They live forever and their permanent identities are determined by their roles in the DM.

An individual buyer has preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t [-H_t + u(x_t)],$$

where H_t is the labor supply in the CM production and x_t is consumption of DM goods. We assume that $u(\cdot)$ is strictly increasing, strictly concave, and twice continuously differentiable with $u'(0) = \infty, u'(\infty) = 0$, and $-x \cdot u''(x)/u'(x) < 1$ for all $x \geq 0$. In addition, let x^* denote the first-best of production in the DM, defined by $u'(x^*) = 1$.

An individual seller has preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t (X_t - h_t),$$

where X_t is consumption of CM goods, and h_t is the labor supply in DM production. Sellers want to consume but cannot produce, whereas buyers can produce but do not want to consume in the CM. Both buyers and sellers use a linear technology in production: one unit of labor is converted into one unit of CM or DM goods.

In the DM, pairwise meetings occur between buyers and sellers. The DM meetings are characterized by frictions such as anonymity and limited commitment among agents. These frictions make a MOE necessary in DM exchanges. Two types of central bank-issued currencies play a role as a MOE: cash and CBDC. Terms of trade are determined through bargaining. For simplicity, we assume that each buyer makes a take-it-or-leave-it offer to his/her counterpart seller.[6] A fraction ρ of meetings in the DM are not monitored, but the rest fraction $1-\rho$ of meetings are monitored by the fiscal authority. In monitored meetings, the fiscal authority imposes a proportional sales tax, $\tau \in [0,1)$, regardless of MOE type.[7] On the other hand, in non-monitored meetings, imposition of sales tax depends on the MOE type. When buyers use CBDC to pay for goods, the fiscal authority can impose sales tax on the CBDC transaction. This is because CBDC transactions are recorded on a digital ledger that the central bank maintains and so transaction information on the ledger can be shared with the fiscal authority. However, when buyers pay cash in exchange for goods, the fiscal authority cannot impose tax because of its anonymity, and thus tax evasion occurs. We assume that if a sales tax is imposed, sellers pay taxes in the next CM, but sellers do not report cash transactions in the non-monitored meetings to the fiscal authority.[8]

In the CM, a buyer knows whether he/she will participate in a monitored meeting or not, and chooses to hold either of two currencies, cash and CBDC, or both for transactions in the forthcoming DM. The superscript $j \in \{n, m\}$ implies non-monitored and monitored meetings, respectively. The stocks of cash (C_t) and CBDC (D_t), which are engineered by the central bank, evolve over time, according to $C_t = \mu_c C_{t-1}$ and $D_t = \mu_d D_{t-1}$, respectively. Let ϕ_t denote the price of cash, and ψ_t denote the price of CBDC in terms of CM goods in period t. As in this type of monetary model, we assume that $\mu_i \geq \beta$ for $i \in \{c, d\}$ for existence of monetary equilibria.

The government, which is the fiscal authority, can collect a proportional sales tax, τ,

[6] We can consider other ways to divide the surplus from trade: for example, proportional bargaining, Nash bargaining, Walrasian price taking, or competitive price posting. As mentioned in Williamson (2012), "given that the seller's utility is linear in labor supply, take-it-or-leave-it offers by the buyer are equivalent to competitive pricing. Take-it-or-leave-it lends tractability to the problem, and avoids distractions associated with determining how the surplus from trade is split." Rocheteau and Wright (2005) and Aruoba et al. (2007) compare different bargaining protocols. Also, see Nosal and Rocheteau (2011) for more details about various bargaining protocols.

[7] Unlike in Vegh (1989) and Aizenman (1983), the tax collection system is assumed to be efficient in our model in the sense that levying a tax does not entail any other costs.

[8] One could argue that most of cash transactions are reported. However, there exists a large amount of literature that studies the shadow, or underground, economy in which tax evasion occurs and presents that its size is not small. For example, according to Gomis-Porqueras et al. (2014), "Estimates for the shadow economy in OECD countries range from 5% of official GDP to 27% while developing economies are much higher, ranging from 25% of official GDP to around 70%." Moreover, since the amount of tax evasion in cash transactions is not critical to derive the most interesting results of this paper, we use the simplest setup here by assuming that none of the cash transactions are reported to the government and that the sales tax is not imposed on these transactions in non-monitored meetings.

from CBDC transactions in non-monitored meetings and from all transactions in monitored meetings, and can receive seigniorage income from the central bank, T_t, to finance its expenditure, G_t. The government provides public goods such as national defense and social infrastructure. Then the government budget constraint is presented by

$$G_t = \rho \tau x_t^n \mathbb{I}_{\{d_t^n > 0\}} + (1-\rho)\tau x_t^m + T_t, \tag{1}$$

where the first term is sales tax revenue in non-monitored meetings, and the second term is sales tax revenue in monitored meetings. x_t^n and x_t^m represent consumption in non-monitored and monitored meetings, respectively. Note that there is an indicator function, $\mathbb{I}_{\{d_t^n>0\}}$, for sales tax revenue in non-monitored meetings because the sales tax can be collected only when CBDC is used as a MOE. If cash is used as a MOE, tax evasion occurs. The sales tax is levied for the real value of goods traded in the DM, and sellers pay the tax in the next CM.

The central bank budget constraints at the beginning of time, i.e., when $t = 0$, and for time $t = 1, 2, ...$ are given by

$$T_0 \leq S_0 = \phi_0 C_0 + \psi_0 D_0,$$

and

$$T_t \leq S_t = \phi_t \left(C_t - C_{t-1}\right) + \psi_t \left(D_t - D_{t-1}\right), \tag{2}$$

where the first term on the right-hand side is seigniorage from cash, and the second term is seigniorage from CBDC. The real value of a lump sum transfer from the central bank to the fiscal authority T_t must be smaller than or equal to the total seigniorage revenue, S_t, for every period. We assume that the central bank makes a strictly positive seigniorage revenue at the initial period, $S_0 > 0$, and it is sufficiently small to create a scarcity of assets in the model. Moreover, since the central bank is not allowed to levy sales tax, we assume that the total seigniorage revenue must be non-negative, $S_t \geq 0$, for time $t = 1, 2, ...$

Lastly, Figure 1 summarizes the events that occur within each period.

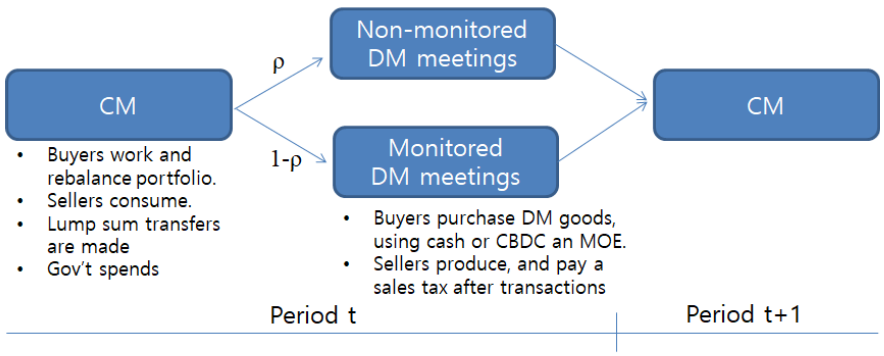

Figure 1: Market Timing

3 Equilibrium

3.1 Individual Agent's Problem

At the beginning of the CM, buyers recognize in which type of meetings they will trade in the next DM, and then choose a currency portfolio of cash and CBDC as a MOE for the next DM transactions. The portfolio is expressed as (c_t^j, d_t^j) for $j \in \{n, m\}$ in terms of CM goods. First, consider buyers who will be in non-monitored meetings. They choose (c_t^n, d_t^n) to consume DM goods, x_t^n. Given the take-it-or-leave-it offers by buyers, the maximization problem is as follows.

$$\max_{x_t^n, c_t^n, d_t^n} u(x_t^n) - c_t^n - (1+\tau)d_t^n, \tag{3}$$

subject to the seller's participation constraint given by

$$\frac{\beta\phi_{t+1}}{\phi_t}c_t^n + \frac{\beta\psi_{t+1}}{\psi_t}d_t^n - x_t^n \geq 0, \tag{4}$$

$x_t^n \geq 0$, $c_t^n \geq 0$ and $d_t^n \geq 0$. Note that sales tax is imposed for transactions in the non-monitored meetings if CBDC is used as a MOE, but not if cash is used.

Buyers who will transact in monitored meetings in the next DM choose (c_t^m, d_t^m) to consume DM goods, x_t^m. Then, the maximization problem is as follows.

$$\max_{x_t^m, c_t^m, d_t^m} u(x_t^m) - (1+\tau)c_t^m - (1+\tau)d_t^m, \tag{5}$$

subject to the seller's participation constraint given by

$$\frac{\beta\phi_{t+1}}{\phi_t}c_t^m + \frac{\beta\psi_{t+1}}{\psi_t}d_t^m - x_t^m \geq 0, \tag{6}$$

$x_t^m \geq 0$, $c_t^m \geq 0$ and $d_t^m \geq 0$.

The following lemma summarizes the solutions to the above maximization problems.

Lemma 1 *Given the rates of return on cash and CBDC, $\frac{\phi_{t+1}}{\phi_t}$ and $\frac{\psi_{t+1}}{\psi_t}$, and the sales tax rate, τ, the consumption and currency portfolio choices of a representative buyer who will trasact in non-monitored meetings are given by*

i) If $\frac{1}{(1+\tau)}\frac{\psi_{t+1}}{\psi_t} < \frac{\phi_{t+1}}{\phi_t}$, then $x_t^n = f\left(\frac{1}{\beta}\frac{\phi_t}{\phi_{t+1}}\right)$, $c_t^n = \frac{1}{\beta}\frac{\phi_t}{\phi_{t+1}}f\left(\frac{1}{\beta}\frac{\phi_t}{\phi_{t+1}}\right)$, and $d_t^n = 0$;

ii) If $\frac{1}{(1+\tau)}\frac{\psi_{t+1}}{\psi_t} > \frac{\phi_{t+1}}{\phi_t}$, then $x_t^n = f\left(\frac{(1+\tau)}{\beta}\frac{\psi_t}{\psi_{t+1}}\right)$, $c_t^n = 0$, and $d_t^n = \frac{1}{\beta}\frac{\psi_t}{\psi_{t+1}}f\left(\frac{(1+\tau)}{\beta}\frac{\psi_t}{\psi_{t+1}}\right)$;

iii) If $\frac{1}{(1+\tau)}\frac{\psi_{t+1}}{\psi_t} = \frac{\phi_{t+1}}{\phi_t}$, then $x_t^n = f\left(\frac{(1+\tau)}{\beta}\frac{\psi_t}{\psi_{t+1}}\right) = f\left(\frac{1}{\beta}\frac{\phi_t}{\phi_{t+1}}\right)$, and portfolio (c_t^n, d_t^n) satisfies the seller's participation constraint (4), but each component within the portfolio is indeterminate.

Here, $f(\cdot) \equiv u'^{-1}(\cdot)$. Next, the consumption and currency portfolio choices of a representative buyer who will trasact in monitored meetings are given by

i) If $\frac{\psi_{t+1}}{\psi_t} < \frac{\phi_{t+1}}{\phi_t}$, then $x_t^m = f\left(\frac{(1+\tau)}{\beta}\frac{\phi_t}{\phi_{t+1}}\right)$, $c_t^m = \frac{1}{\beta}\frac{\phi_t}{\phi_{t+1}}f\left(\frac{(1+\tau)}{\beta}\frac{\phi_t}{\phi_{t+1}}\right)$, and $d_t^m = 0$;

ii) If $\frac{\psi_{t+1}}{\psi_t} > \frac{\phi_{t+1}}{\phi_t}$, then $x_t^m = f\left(\frac{(1+\tau)}{\beta}\frac{\psi_t}{\psi_{t+1}}\right)$, $c_t^m = 0$, and $d_t^m = \frac{1}{\beta}\frac{\psi_t}{\psi_{t+1}}f\left(\frac{(1+\tau)}{\beta}\frac{\psi_t}{\psi_{t+1}}\right)$;

iii) If $\frac{\psi_{t+1}}{\psi_t} = \frac{\phi_{t+1}}{\phi_t}$, then $x_t^m = f\left(\frac{(1+\tau)}{\beta}\frac{\psi_t}{\psi_{t+1}}\right) = f\left(\frac{1}{\beta}\frac{\phi_t}{\phi_{t+1}}\right)$, and portfolio (c_t^m, d_t^m) satisfies the seller's participation constraint (6), but each component within the portfolio is indeterminate.

Proof. See the appendix. ■

Lemma 1 has an intuitive interpretation. The portfolio choices of buyers depend on the relative rates of return on cash and CBDC. Notice that in non-monitored meetings, the rate of return on CBDC allows for the sales tax rate, τ, unlike the one on cash. If the real rate of return on CBDC after tax, $\frac{1}{(1+\tau)}\frac{\psi_{t+1}}{\psi_t}$, is lower (higher) than that on cash, $\frac{\phi_{t+1}}{\phi_t}$, buyers prefer to use cash (CBDC) as a MOE in transactions because cash allows more consumption of DM goods. For instance, consumption x_t^n in non-monitored meetings is determined by $f(\cdot)$. It is obvious that, when $\frac{1}{(1+\tau)}\frac{\psi_{t+1}}{\psi_t} < \frac{\phi_{t+1}}{\phi_t}$, $f\left(\frac{(1+\tau)}{\beta}\frac{\psi_t}{\psi_{t+1}}\right) < f\left(\frac{1}{\beta}\frac{\phi_t}{\phi_{t+1}}\right)$ since f is a strictly decreasing function. Unlike non-monitored meetings, both cash and CBDC transactions pay the sales tax in monitored meetings, and thus buyers' choices depend on the relative rates of return on cash and CBDC themselves without considering sales tax rate. If the real rate of return on CBDC, $\frac{\psi_{t+1}}{\psi_t}$, is lower (higher) than that on cash, $\frac{\phi_{t+1}}{\phi_t}$, buyers prefer to use cash (CBDC) as a MOE in transactions for the similar reason as above. Lastly, if the rates of return on cash and CBDC are equal, then buyers are indifferent to using either as media of exchange, and thus hold only one or both of the two currencies. In this case, buyers consume the same amount of DM goods, irrespective of MOE type.

3.2 Stationary Equilibrium

We focus on steady-state equilibria in which real quantities are constant over time, and thus the real balances of cash and CBDC are also constant. This implies that

$$\phi_t C_t = \phi_{t+1} C_{t+1} \text{ and } \psi_t D_t = \psi_{t+1} D_{t+1}. \tag{7}$$

Then, $\frac{\phi_t}{\phi_{t+1}} = \mu_c$ and $\frac{\psi_t}{\psi_{t+1}} = \mu_d$ hold for all time t in the steady state. Then we define a stationary equilibrium as follows.

Definition 1 *A stationary equilibrium consists of a list $\{x^m, c^m, d^m, x^n, c^n, d^n\}$ such that*

(i) the decision rule of a representative buyer solves the individual optimization problem (3)-(5), taking prices $\phi_t/\phi_{t+1} = \mu_c$, $\psi_t/\psi_{t+1} = \mu_d$, the government spending, G, and the sales tax rate τ as given;

(ii) Markets clear and expectations are rational as follows.

$$\rho c^n + (1-\rho)(1+\tau)c^m = \phi_t C_t, \tag{8}$$

$$\rho(1+\tau)d^n + (1-\rho)(1+\tau)d^m = \psi_t D_t, ; \tag{9}$$

for all $t \geq 1$.

(iii) the government and central bank budget constraints, (1) and (2), hold respectively, or

$$G = \rho\tau\frac{\beta}{\mu_d}d^n + (1-\rho)\tau\left(\frac{\beta}{\mu_c}c^m + \frac{\beta}{\mu_d}d^m\right) + T, \tag{10}$$

$$T \leq S = (1-\frac{1}{\mu_c})\left[\rho c^n + (1-\rho)(1+\tau)c^m\right] + (1-\frac{1}{\mu_d})\left[\rho(1+\tau)d^n + (1-\rho)(1+\tau)d^m\right]. \tag{11}$$

It is worth noticing how the sales tax rate can affect tax revenue for government spending. For example, consider an equilibrium in which only cash is used in non-monitored meetings and only CBDC used in monitored meetings: $c^n > 0$, $d^n = 0$, $c^m = 0$ and $d^m > 0$. If τ increases, the sales tax revenue increases directly by $(1-\rho)\tau\frac{\beta}{\mu_d}d^m$ in (10) and seigniorage increases by $(1-\frac{1}{\mu_d})(1-\rho)\tau d^m$ in (11). However, if the rate of return on CBDC needs to be maintained for a constant demand for CBDC, d^m, then raising the sales tax rate may not always increase the entire tax revenue for government spending. When τ increases, μ_d must decrease to keep $(1+\tau)\mu_d$ constant. As a result, sale tax revenue, $(1-\rho)\tau\frac{\beta}{\mu_d}d^m$, will certainly increase, but seigniorage revenue, $(1-\frac{1}{\mu_d})(1-\rho)\tau d^m$, may decrease. Although the real quantity for monitored trades, $(1-\rho)d^m$, increases, the net rate of return for seigniorage, $1-\frac{1}{\mu_d}$, decreases.

3.3 Types of Equilibria

Now we characterize equilibria. Given the equilibrium conditions, if $\tau > 0$, there exist five different types of equilibria that are feasible for a monetary policy combination of (μ_c, μ_d). Table 1 shows those equilibria.

Table 1: Possible Equilibria

	Monitored	Cash only ($\mu_c < \mu_d$)	CBDC only ($\mu_c > \mu_d$)	Both ($\mu_c = \mu_d$)
	Cash only ($\mu_c < \mu_d(1+\tau)$)	O (3.3.1)	O (3.3.3)	O (3.3.2)
Non-monitored	CBDC only ($\mu_c > \mu_d(1+\tau)$)	X	O (3.3.1)	X
	Both ($\mu_c = \mu_d(1+\tau)$)	X	O (3.3.2)	X

In Table 1, an 'X' denotes an equilibrium that is not feasible, whereas an 'O' denotes an equilibrium that is feasible. Which equilibrium arises depends on the relative rates of return on cash and CBDC, that is, $\frac{1}{\mu_c}$ and $\frac{1}{\mu_d(1+\tau)}$ in non-monitored meetings and $\frac{1}{\mu_c}$ and $\frac{1}{\mu_d}$ in monitored meetings.

3.3.1 Pooling Equilibrium: $\mu_c < \mu_d$ or $\mu_c > \mu_d(1+\tau)$

When $\mu_c < \mu_d$ or $\mu_d(1+\tau) < \mu_c$ holds, only one of cash and CBDC is used as a MOE in both types of meetings in equilibrium, and so it obviously gives rise to results that we could find in a one-currency economy. In the former case, the relative rate of return on cash is higher than that of CBDC, irrespective of the sales tax rate, τ, and thus only cash is used in both monitored and non-monitored transactions, whereas in the latter case, only CBDC is used in both transaction types for a similar reason.

3.3.2 Partially Pooling Equilibrium: $\mu_c = \mu_d$ or $\mu_c = \mu_d(1+\tau)$

First, consider a case in which $\mu_c = \mu_d$ holds. If the sales tax is equal to zero, i.e., $\tau = 0$, then both cash and CBDC are used as a MOE in both monitored and non-monitored meetings. There is no difference between cash and CBDC as a MOE in the sense that they yield the same real rates of return. However, if $\tau > 0$, that is, $\mu_c = \mu_d < \mu_d(1+\tau)$, either of cash and CBDC, or both are used in monitored meetings, and only cash is used in non-monitored meetings. When transactions are monitored, there is no difference between the two media of exchange because their rates of return are equal. However, when transactions are not monitored, the rate of return on cash is higher than that on CBDC because buyers can avoid sales tax. As a result, cash as a MOE allows buyers to consume more than does CBDC in non-monitored meetings.

Next, consider a case in which $\mu_c = \mu_d(1+\tau)$. If $\tau = 0$, it is obvious that the same equilibrium occurs as the one in which $\mu_c = \mu_d$. However, if $\tau > 0$, $\mu_c > \mu_d$. Only CBDC is used as a MOE in monitored meetings, whereas both cash and CBDC, or one of them can be used in non-monitored meetings. This is because the real rate of return on CBDC is higher than that on cash, but its rate of return after tax is the same as that on cash.

3.3.3 Separating Equilibrium: $\mu_d < \mu_c < (1+\tau)\mu_d$

Note that the sales tax is strictly positive, $\tau > 0$, for this equilibrium to arise. In non-monitored meetings, the rate of return on cash, $\frac{1}{\mu_c}$, is higher than that on CBDC, $\frac{1}{(1+\tau)\mu_d}$, because the sales tax is collected only with CBDC. On the other hand, in monitored meetings in which sales tax is imposed on all transactions, the rate of return on CBDC, $\frac{1}{(1+\tau)\mu_d}$, is higher than that on cash, $\frac{1}{(1+\tau)\mu_c}$. As a result, only cash is used in non-monitored meetings whereas only CBDC is used in monitored meetings, i.e., $d^m > 0$, $c^n > 0$ and $c^m = d^n = 0$ in equilibrium. In this equilibrium the central bank can flexibly choose the rates of return on the two monies separately, as long as the policy variables (τ, μ_c, μ_d) satisfy the central bank and government budget constraints, (10) and (11).

3.4 Sales and Inflation Tax Revenue

Before we move on to the welfare analysis in the next section, it is worth examining how the sales tax rate, τ, and inflation tax rate, μ_j for $j \in \{c, d\}$, affect the government's sales tax revenue and the central bank's seigniorage revenue.

The sales tax revenue, denoted by T^s, is collected in monitored meetings or in non-monitored meetings when CBDC is used as a MOE. We rewrite the equation for sales tax revenue, which is represented by the first two terms on the right-hand side of (10), using the solutions in Lemma 1 as below.

$$T^s \equiv \rho\tau f\left(\frac{\mu_d(1+\tau)}{\beta}\right) + (1-\rho)\tau f\left(\frac{\mu_j(1+\tau)}{\beta}\right) \tag{12}$$

for $j \in \{c, d\}$. Since demand for a MOE, $f(\frac{\mu_j(1+\tau)}{\beta})$ for $j \in \{c, d\}$, is negatively related to tax rate, τ, in both types of meetings, we can find out a *Laffer* curve for sales tax collection as presented in the following lemma.

Lemma 2 *Given μ_j for $j \in \{c, d\}$, $\frac{\partial T^s}{\partial \tau} > 0$ holds in $\tau \in [0, \bar{\tau})$.*

Proof. See the appendix. ■

Similarly, the inflation tax, i.e. the seigniorage, can be collected in both types of meetings, so we can rewrite the central bank constraint (11) using the solutions in Lemma 1 as below.

$$S = \rho\left(\frac{\mu_i - 1}{\beta}\right) f\left(\frac{\mu_i}{\beta}\right) + (1-\rho)\left(\frac{\mu_j - 1}{\beta}\right)(1+\tau) f\left(\frac{\mu_j(1+\tau)}{\beta}\right) \geq 0 \tag{13}$$

for $i, j \in \{c, d\}$. Inflation tax revenue, S, in (11) is also hump-shaped in μ_i, because the demand for media of exchange, $f(\frac{\mu_i}{\beta})$ and $f(\frac{\mu_j(1+\tau)}{\beta})$, increase in their rates of return, $\frac{1}{\mu_i}$ and $\frac{1}{\mu_j}$, respectively.

Lemma 3 *Given τ and μ_j, $\frac{\partial S}{\partial \mu_i} > 0$ holds in $\mu_i \in [\beta, \bar{\mu}_i)$. Similarly, given τ and μ_i, $\frac{\partial S}{\partial \mu_j} > 0$ holds in $\mu_j \in [\beta, \bar{\mu}_j)$.*

Proof. See the appendix. ■

It is straightforward to show that if $\mu_i = \mu_j$, $1 \leq \mu_i$ must hold in (13) with equality. Also, if $\mu_i > \mu_j$, then $\beta \leq \mu_j < 1 < \mu_i$ is feasible for $S = 0$. Intuitively, the central bank can allow a loss in seigniorage from one currency as long as the other currency provides more seigniorage.

Hereafter, we focus solely on equilibrium allocations in which the sales tax revenue increases in the tax rate and the seigniorage decreases in the rate of return on a MOE, because it is inefficient to impose higher sales tax rates and inflation taxes to finance a certain amount of government expenditures. Lastly, we assume that the required government expenditure, $G > 0$, is feasible at $\tau \in [0, \bar{\tau})$, $\mu_i \in [\beta, \bar{\mu}_i)$ and $\mu_i \in [\beta, \bar{\mu}_j)$ for $i, j = \{c, d\}$.

4 Welfare Comparisons

We investigate whether introducing CBDC is beneficial for welfare or not, by comparing the equilibrium allocations that can be achieved without and with CBDC. Furthermore, we examine how CBI affects this comparison. We define the aggregate welfare by adding the expected utility across the agents in a stationary equilibrium as

$$W = \rho\{u(x^n) - x^n\} + (1-\rho)\{u(x^m) - x^m\}. \tag{14}$$

We consider an economy in which the fiscal authority chooses the sales tax rate, τ, to support government expenditure, G, passively, given monetary policy (μ_c, μ_d). Then, given this fiscal policy τ, the central bank chooses (μ_c, μ_d) to maximize welfare. In this respect, the central bank plays the role of planner in this economy. We also define an additional term, ζ, by setting $\zeta = \frac{\mu^d}{\mu^c}$ in order to illustrate the relative rate of return between the media of exchange. Then, $\frac{1}{\zeta} - 1$ can be interpreted as the interest rate on CBDC, which can be either positive or negative. Hence, the central bank can choose ζ instead of μ_d.

4.1 Without the Central Bank Independence

Suppose that CBI is not required: a transfer from the central bank to the fiscal authority can be strictly positive. Since the seigniorage left in the central bank is useless, the central bank constraint always binds with $T = S$. From (10)-(13), the consolidated government budget constraint can be shown as

$$G = \rho\left(\frac{\mu_c - 1}{\beta}\right) f\left(\frac{\mu_c}{\beta}\right) + (1-\rho)\left(\frac{\zeta\mu_c - 1}{\beta}\right)(1+\tau) f\left(\frac{\zeta\mu_c(1+\tau)}{\beta}\right) + (1-\rho)\tau f\left(\frac{\zeta\mu_c(1+\tau)}{\beta}\right), \tag{15}$$

when cash is used in non-monitored meetings. This representation has an advantage to compare the allocations of all the equilibria only except for the pooling equilibrium with CBDC. Since the sales tax is required with either cash or CBDC in monitored meetings, we can consider $\zeta = 1$ as the case that only cash is used, and $\zeta < 1$ as the case that cash is used in non-monitored meetings while CBDC is used in monitored meetings. We will analyze the case that only CBDC is used in the following subsection separately. Note that, if the consolidated authority chooses ζ and τ, μ_c is determined by this consolidated budget constraint in equilibrium.

Using the solutions in Lemma 1, we can rewrite the consolidated government budget constraint (15) as

$$G = \rho\left(u'(x^n) - \frac{1}{\beta}\right)x^n + (1-\rho)\left(u'(x^m) - \frac{1}{\beta}\right)x^m - (1-\rho)\frac{1}{\beta}\tau x^m + (1-\rho)\tau x^m, \tag{16}$$

where $\tau = \frac{u'(x^m)}{\zeta u'(x^n)} - 1$ by $u'(x^m) = \frac{\mu_c(1+\tau)}{\beta}$ and $u'(x^n) = \frac{\mu_c}{\beta}$ in Lemma 1. The first term in

(16) is the seigniorage from non-monitored meetings, and the sum of the second and third terms is the seigniorage from monitored meetings. The fourth term represents the sales tax revenue. Notice that the third term represents a loss in the seigniorage caused by raising the sales tax rate. Raising the sales tax rate reduces the currency demand for transactions, but the rate of return on seigniorage for each currency demand is maintained. As long as $\tau > 0$ in (16), the sum of the third and fourth terms is negative: the loss in the seigniorage always exceeds the additional sale tax revenue. The reason can be explained as follows. Inflation tax is collected when currency is issued whereas sales tax is collected when currency is used. Since there exists a time difference between the point of time in which currency is held by the buyers and the point of time where currency is used up, inflation tax can be collected more than sales tax even if they reduce the currency demand to the same extent.[9] Thus, sales tax is less efficient than inflation tax to support a certain amount of government expenditure.[10]

Finally, note that given τ, there is a strict negative relationship between x^n and x^m in (16). Since the seigniorage revenue increases in μ_c (or $\zeta\mu_c$) by Lemma 3, $\{(u'(x^k) - \frac{1}{\beta}\}x^k$ decreases in x^k for $k \in \{n, m\}$.

4.1.1 Cash-only economy without CBI

If the rate of return on CBDC is lower than that on cash, $\mu_c < \mu_d$, only cash will be used in both meetings, which is a pooling equilibrium. In this case we can apply a condition, $\zeta = 1$, into the equilibrium conditions. When the consolidated authority chooses a target τ, μ_c is determined by the consolidated budget constraint (15).

Proposition 1 *In a cash-only economy without CBI, the optimal policy is $\tau = 0$ and $\mu_c = \hat{\mu}_c > 1$.*

Proof. See the appendix. ■

The allocation achieved by $\tau = 0$ is described as point A in Figure 2.[11] Given $\zeta = 1$, the first-order condition curve, $FOC_{\tau=0}$, is shown as a 45-degree line because $x^n = x^m$.

[9] Although the sub-periods in this model begins with DM instead of CM, this result does not change. Inflation tax is collected at the CM in period $t-1$, while sales tax is collected at the CM in period t.

[10] This property can be also shown in the other monetary models such as cash-in-advance and over-lapping generation models as long as currency must be held one period before to consume.

[11] In Figure 2, point FR represents the Friedman Rule allocation, which is implemented by $\mu_c = \mu_d = \beta$.

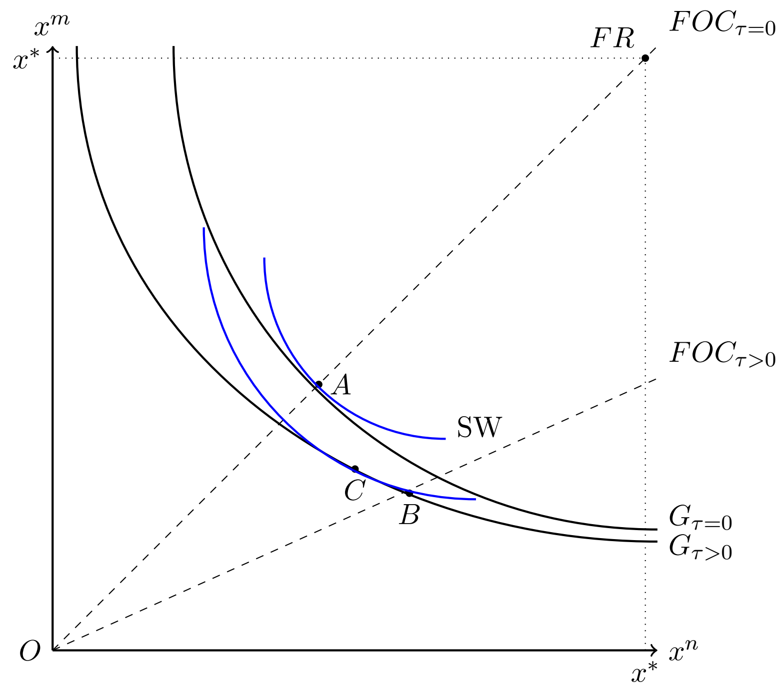

Figure 2: Equilibrium allocations without CBI

The consolidated government budget constraint, indicated by $G_{\tau=0}$, is shown as symmetric around the 45 degree line when $\tau = 0$. At point A, a social welfare curve, indicated by a blue curve SW, is tangential to the consolidated government budget constraint, $G_{\tau=0}$.

Now consider a case in which $\tau > 0$ to examine the distortionary effect that a positive sales tax causes. Suppose that τ increases. The FOC curve rotates clockwise as x^n increases and x^m decreases.[12] Moreover, if $\tau > 0$, the G curve moves toward the origin and rotates counterclockwise. Since the sales tax is less efficient than the inflation tax to support the government spending, there arises a loss in the feasibility set. Additionally, levying the sales tax leads to decrease the consumption in monitored meetings more than the consumption in non-monitored meetings, because the sales tax does not apply for the cash transactions in non-monitored meetings. The new equilibrium allocation is located at point B, at which the welfare is lower than that at point A. This is obvious because point B is feasible with the government budget constraint at $\tau = 0$. Note that there arise two types of inefficiencies

[12]It is worth pointing out that our results are consistent with the optimal taxation principle discussed in Lucas and Stokey (1983) and Lucas (1990). They find that it is optimal to levy tax on the commodities that enter consumer utility symmetrically at the same rate.

associated with the strictly positive sales tax, $\tau > 0$. One is the distortion of the relative marginal utility between non-monitored and monitored meetings. For example, at point B the social welfare curve is not tangential to the consolidated government budget constraint, $G_{\tau>0}$, unlike at point C. The other inefficiency is the loss in the feasibility because we cannot achieve the allocation at point A with $G_{\tau>0}$.

4.1.2 Cash and CBDC economy without CBI

Suppose that the consolidated authority introduces CBDC into the economy. It is obvious that the optimal allocation in a cash-only economy can be easily replicated by choosing $\tau = 0$ and $\zeta = 1$ (or $\mu_d = \mu_c = \hat{\mu}_c$). In this case, buyers are indifferent to using either cash or CBDC, or both in monitored and non-monitored meetings, which is a partially pooling equilibrium. The consumption levels in both types of meetings are equal: $x^n = x^m$. The following proposition presents that the optimal policy mix is the same as that in a cash-only economy.

Proposition 2 *In a cash and CBDC economy without CBI, the optimal policy mix is* $\tau = 0$, $\zeta = 1$ *and* $\mu_c = \hat{\mu}_c > 1$.

Proof. See the appendix. ■

Notice that the optimal allocation in a cash-only economy is unique and can be replicated only at $\tau = 0$. As we discussed, when $\tau > 0$, the government budget constraint, G, moves toward the origin and rotates counterclockwise. Given $\tau > 0$, the consolidated government can reduce ζ to move the equilibrium allocation to the top left toward point A in Figure 2, which provides higher welfare than that at point B. $\zeta < 1$ and $\tau > 0$ imply that the central bank collects more inflation tax from buyers who use cash to avoid sales tax, and provides a transfer to the buyers who pay sales tax in monitored meetings. This implicit transfer can improve welfare because it can fix the inefficiency associated with the relative marginal utility between non-monitored and monitored meetings. Nevertheless, it is inevitable for a positive sales tax rate to generate a loss in the feasibility, because sales tax is less efficient than inflation tax. Proposition 2 shows that CBDC is redundant in the economy without CBI in the sense that it cannot expand the feasible allocation set than in a cash-only economy.

4.1.3 CBDC-only economy without CBI

Suppose that the consolidated authority eliminates cash from the economy. Buyers can use only CBDC in both types of meetings, and so buyers in non-monitored meetings are also subject to sales tax. Note that the government can still pay positive(or negative) interest to one of the two types of meetings. The consolidated government budget constraint can be described as

$$\begin{aligned} G = &\rho\left(\frac{\mu_d - 1}{\beta}\right)(1+\tau)f\left(\frac{\mu_d(1+\tau)}{\beta}\right) + \rho\tau f\left(\frac{\mu_d(1+\tau)}{\beta}\right) \\ &+ (1-\rho)\left(\frac{\zeta\mu_d - 1}{\beta}\right)(1+\tau)f\left(\frac{\zeta\mu_d(1+\tau)}{\beta}\right) + (1-\rho)\tau f\left(\frac{\zeta\mu_d(1+\tau)}{\beta}\right), \end{aligned} \tag{17}$$

and we can transform it again into

$$G = \rho\left(u^{'}(x^n) - \frac{1}{\beta}\right)x^n - \rho(\frac{1}{\beta} - 1)\tau x^n + (1-\rho)\left(u^{'}(x^m) - \frac{1}{\beta}\right)x^m - (1-\rho)(\frac{1}{\beta} - 1)\tau x^m. \tag{18}$$

Proposition 3 *In a CBDC only economy without the CBI, the optimal policy is* $\tau = 0$, $\zeta = 1$ *and* $\mu_d = \hat{\mu}_c > 1$.

Proof. See the appendix. ■

Proposition 3 presents that eliminating cash cannot improve welfare in the economy without CBI. Since sales tax is less efficient than inflation tax, the consolidated government will finance its spending by collecting seigniorage. However, CBDC with record-keeping technology does not help in expanding the feasible allocation set.

4.2 With Central Bank Independence

We define central bank independence as $T \leq 0$: a strictly positive transfer from the central bank to the fiscal authority is not allowed. Given $G > 0$, since it is not efficient to collect sales tax to make a transfer from the fiscal authority to the central bank, the transfer must be zero, $T = 0$, in equilibrium. Thus, the government spending, G, is supported only by the sales tax as follows.

$$G = T^s = (1-\rho)\tau f\left(\frac{\zeta\mu_c(1+\tau)}{\beta}\right), \tag{19}$$

and the non-negative seigniorage condition is given by

$$S = \rho\left(\frac{\mu_c - 1}{\beta}\right) f\left(\frac{\mu_c}{\beta}\right) + (1-\rho)\left(\frac{\zeta\mu_c - 1}{\beta}\right)(1+\tau) f\left(\frac{\zeta\mu_c(1+\tau)}{\beta}\right) \geq 0. \tag{20}$$

Note that the sales tax rate, τ, must be strictly positive for $G > 0$ in (19), and thus the optimal allocation in an economy without CBI cannot be achieved in an economy with CBI. Also, in equilibrium, (20) must bind with equality, because if $S > 0$ holds, reducing μ_c can raise both x^n and x^m, as seigniorage revenue decreases. Consequently, if the central bank chooses ζ and μ_c to satisfy (20) with equality, τ is determined by (19).

The following lemma presents that the relationship between the sales tax rate and the rate of return on a MOE used in meetings which pay tax. In order to finance a certain level of G, a higher tax rate leads to a lower rate of return; otherwise, sales tax revenue declines because the trade volume decreases.

Lemma 4 *When the CBI is required and G is strictly positive, τ and $\zeta\mu_c$ are positively related in equilibrium.*

Proof. See the appendix. ■

4.2.1 Cash-only economy with CBI

As in the previous subsection, we apply a condition, $\zeta = 1$, into the equilibrium conditions (19) and (20). If the central bank chooses μ_c, then τ is determined by (19). Given this fiscal policy, τ, the central bank will reduce μ_c until (20) binds at zero, i.e. $S = 0$ to increase both x^n and x^m. Since a lower μ_c is associated with a lower τ by Lemma 4, the lowest μ_c maximizes welfare.

Proposition 4 *In a cash-only economy with CBI, the optimal monetary and fiscal policy mix is $\mu_c = 1$, and $\tau = \hat{\tau} > 0$, where $\hat{\tau}$ solves $G = (1-\rho)\hat{\tau} f(\frac{1+\hat{\tau}}{\beta})$.*

Proof. See the appendix. ■

In this case, both types of distortion that a strictly positive τ can cause arise. Since we have only one MOE, the marginal utilities in non-monitored and monitored meetings cannot be equal with $\tau > 0$. Moreover, the feasible allocation set shrinks, which is described as the curve $G = T^s(\tau > 0)$ in Figure 3.

4.2.2 Cash and CBDC economy with CBI

Now, suppose that the central bank introduces CBDC into the economy. If $\zeta = 1$ and $\mu_c = 1$ are chosen, the optimal allocation in the cash-only economy with CBI can be replicated. Hence, we focus on whether the welfare of the optimal allocation in this cash and CBDC economy with CBI can be greater than that in the cash-only economy with CBI or not.

Let us define $\tilde{\mu} = \zeta\mu_c(1+\tau)$. Then, (20) can be rewritten as

$$\rho\left(\frac{\mu_c - 1}{\beta}\right)f\left(\frac{\mu_c}{\beta}\right) + (1-\rho)\left(\frac{\tilde{\mu}-1}{\beta}\right)f\left(\frac{\tilde{\mu}}{\beta}\right) \geq (1-\rho)\frac{\tau}{\beta}f\left(\frac{\tilde{\mu}}{\beta}\right) = \frac{G}{\beta}, \tag{21}$$

Note that given $\tilde{\mu}$, τ is determined to satisfy (19). Unlike the previous cash-only economy, the central bank can adjust ζ (or $\tilde{\mu}$) additionally to change the relative ratio between μ_c and $\tilde{\mu}$. Using the first-order conditions in Lemma 1, the non-negative seigniorage condition (21) with equality can be transformed into

$$\frac{G}{\beta} = \rho x^n\{u^{'}(x^n) - \frac{1}{\beta}\} + (1-\rho)x^m\{u^{'}(x^m) - \frac{1}{\beta}\}. \tag{22}$$

The following proposition demonstrates the optimal fiscal and monetary policy.

Proposition 5 *In a cash and CBDC economy with CBI, the optimal policy mix is* $\zeta = \frac{1}{1+\tau^*}$, $\mu_c = \mu_c^* > 1$, *and* $0 < \tau^* < \hat{\tau}$. μ_c^* *solves* $G = (\mu_c^*-1)f\left(\frac{\mu_c^*}{\beta}\right)$, *and* τ^* *solves* $G = (1-\rho)\tau^* f(\frac{\mu_c^*}{\beta})$. *The optimal policy with cash and CBDC improves welfare.*[13]

Proof. See the appendix. ■

In Figure 3, we describe the optimal allocation which Proposition 5 presents. Without CBI, the consolidated authority can set $\tau = 0$, and thus point A can be achieved with (22).

[13] In order to guarantee the interior solution in this case, we need to assume that G is sufficiently small and the time preference is sufficiently high as $\beta > 1-\rho$. Since agents can carry CBDC with a positive interest just for arbitrage profit, $\zeta\mu_c = \frac{\mu_c^*}{1+\tau^*} > \beta$ must hold in equilibrium. By using $G = (\mu_c^* - 1)f\left(\frac{\mu_c^*}{\beta}\right) = (1-\rho)\tau^* f\left(\frac{\mu_c^*}{\beta}\right)$, we can transform it into $\mu_c^* < \frac{\beta - \beta(1-\rho)}{\beta-(1-\rho)}$. Since μ_c^* decreases in G, this condition will hold when $\beta > 1-\rho$ and G is sufficiently small.

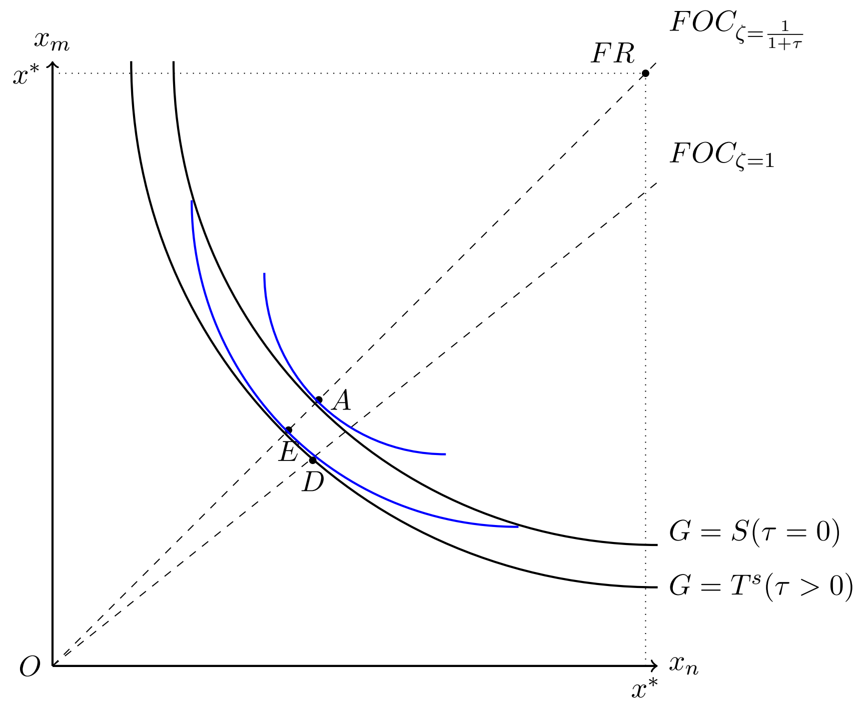

Figure 3: Equilibrium allocations with CBI

However, when CBI is required, imposing a positive sales tax is inevitable to finance G, and the fiscal authority's constraint (22) moves toward the origin symmetrically without any rotation. Since the required sales tax revenue is fixed as G, changes in the sale tax rate is not involved with the distortion of the loss in the feasible allocation set, $\frac{G}{\beta} - G$, any more. In the cash-only economy with CBI, the equilibrium allocation is located at point D in Figure 3, at which $x^n > x^m$ when $\tau > 0$ and $\zeta = 1$. If CBDC is introduced, paying a strictly positive interest, $\frac{1}{\zeta} - 1$, on CBDC can move the equilibrium allocation from point D to point E, at which welfare is higher. Note that the equilibrium allocation in the cash-only economy with CBI is still feasible here. In other words, welfare that can be achieved in a cash and CBDC economy with CBI is greater than that in a cash-only economy with CBI as presented in Proposition 5.

The reason why paying a positive interest on CBDC is welfare-improving is that it removes at least one of the two type of distortion that a positive sales tax causes. Specifically, it can eliminate a distortion in the marginal rates of substitution between non-monitored and monitored transactions by adjusting the relative rates of return on CBDC and cash. A positive interest rate on CBDC increases the relative rate of return on CBDC to that of cash, and thus demand for CBDC as a MOE.[14] This increases consumption in monitored meetings, which can be described as a counterclockwise rotation of the FOC curve in Figure 3. As a result, it equalizes the slopes of government spending, G, and social welfare curves.

4.2.3 CBDC-only economy with CBI

Now, suppose that the central bank eliminates cash from the economy. As we discuss above, buyers in non-monitored meetings are also subject to sales tax because the only MOE is CBDC. Government spending, G, is supported by sales tax revenue from both types of meetings as

$$G = T^s = \rho\tau f\left(\frac{\mu_d(1+\tau)}{\beta}\right) + (1-\rho)\tau f\left(\frac{\zeta\mu_d(1+\tau)}{\beta}\right), \tag{23}$$

and the non-negative seigniorage condition is given by

$$S = \rho\left(\frac{\mu_d - 1}{\beta}\right)(1+\tau)f\left(\frac{\mu_d(1+\tau)}{\beta}\right) + (1-\rho)\left(\frac{\zeta\mu_d - 1}{\beta}\right)(1+\tau)f\left(\frac{\zeta\mu_d(1+\tau)}{\beta}\right) \geq 0. \tag{24}$$

As we derived before, when we define $\tilde{\mu}_d = \mu_d(1+\tau)$, (24) can be rewritten as

$$\rho\left(\frac{\tilde{\mu}_d - 1}{\beta}\right)f\left(\frac{\tilde{\mu}_d}{\beta}\right) + (1-\rho)\left(\frac{\zeta\tilde{\mu}_d - 1}{\beta}\right)f\left(\frac{\zeta\tilde{\mu}_d}{\beta}\right) \geq \rho\frac{\tau}{\beta}f\left(\frac{\tilde{\mu}_d}{\beta}\right) + (1-\rho)\frac{\tau}{\beta}f\left(\frac{\zeta\tilde{\mu}_d}{\beta}\right) = \frac{G}{\beta}, \tag{25}$$

and so the non-negative seigniorage condition (22) in the cash and CBDC economy holds.

Proposition 6 *In a CBDC only economy with CBI, the optimal policy is* $\tau = \tau^{**} < \tau^*$, $\zeta = 1$ *and* $\mu_d = \hat{\mu}_d < \mu_c^*$.

Proof. See the appendix. ■

Proposition 6 demonstrates that eliminating cash cannot improve welfare further, compared with the cash and CBDC economy with CBI. Since all transactions are monitored and recorded, the government can collect sales tax from both types of meetings. However, as long

[14] It is worth noting that this mechanism works even when bank deposits are available as a MOE, in particular through those who cannot access financial services. Furthermore, it could work for those who choose bank deposits as a means of payment in monitored meetings because bank deposits backed by interest-bearing CBDC would yield a higher return than bank deposit backed by cash.

as the required sales tax revenue is fixed as G, the corresponding loss in the feasibility set occurs regardless of who pays the sales tax. The sales tax rate can be lower, compared with the cash and CBDC economy, because the tax burden is equally distributed to all buyers. Instead, the rate of return on CBDC, $\frac{1}{\mu_d}$, increases slightly to equalize the after-tax return as the same as the one in cash and CBDC economy, $\frac{1}{\hat{\mu}_d(1+\tau^{**})} = \frac{1}{\mu_c^*}$. This result implies that if a currency with record-keeping technology can be introduced, it is not necessary to record all transactions because the government can make a transfer between meetings that use the record-keeping currency and meetings that do not. This transfer can remove the distortion in the marginal rates of substitution between the two types of meetings by adjusting the relative rates of return on media of exchange.

5 Conclusion

This paper studies how introducing CBDC affects welfare in an economy where tax evasion occurs in cash transactions. Our key finding is that CBDC is beneficial for welfare when the central bank is independent in the sense that no monetary transfer between the central bank and the fiscal authority is allowed. In the economy without CBI, collecting inflation tax only is the optimal solution to finance a positive level of government spending, G, because inflation tax is more efficient than sales tax in tax collection. In this case, introducing CBDC is not welfare-improving in the sense that the feasible allocation set does not change. On the other hand, in the economy with CBI, a positive sales tax is inevitable, and generates the two types of distortion: losses in tax collection and a misalignment in the relative marginal utility between tax-evaded and tax-payed transactions. Introducing CBDC with a positive interest can eliminate the latter distortion, although the former remains because a certain amount of sales tax must be collected. This expands the feasible allocation set. The positive interest rate on CBDC plays a role of transferring a seigniorage revenue from transactions which evade sales tax to those who pay it, and thus improves welfare by increasing consumption in taxed transactions.

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A Appendix

Proof. Proof of Lemma 1.

First, we set up the Lagrangian function for a buyer who will not be monitored in the next DM as follows.

$$\mathcal{L}^n = u(x^n) - c^n - (1+\tau)d^n + \lambda^n \left(\beta \frac{\phi_{t+1}}{\phi_t} c^n + \beta \frac{\psi_{t+1}}{\psi_t} d^m - x^n \right). \tag{26}$$

Then, the Kuhn-Tucker conditions are

$$\begin{aligned} u'(x^n) \leq \lambda^n; \quad x^n \geq 0; \quad & x^n \left(u'(x^n) - \lambda^n\right) = 0, \\ \lambda^n \beta \frac{\phi_{t+1}}{\phi_t} \leq 1; \quad c^n \geq 0; \quad & c^n \left(\lambda^n \beta \frac{\phi_{t+1}}{\phi_t} - 1 \right) = 0, \\ \lambda^n \beta \frac{\psi_{t+1}}{\psi_t} \leq 1+\tau; \quad d^n \geq 0; \quad & d^n \left(\lambda^n \beta \frac{\psi_{t+1}}{\psi_t} - 1 - \tau \right) = 0. \end{aligned}$$

We have the following first-order condition

$$u'(x^n) = \lambda^n. \tag{27}$$

Given the rates of return on cash and CBDC, $\frac{\phi_{t+1}}{\phi_t}$, and $\frac{\psi_{t+1}}{\psi_t}$, and the sales tax rate, τ, a buyer in the non-monitored DM meeting chooses

$$\begin{aligned} c^n > 0, d^n = 0 \quad & \text{if} \quad \frac{1}{(1+\tau)} \frac{\psi_{t+1}}{\psi_t} < \frac{\phi_{t+1}}{\phi_t}, \\ c^n = 0, d^n > 0 \quad & \text{if} \quad \frac{1}{(1+\tau)} \frac{\psi_{t+1}}{\psi_t} > \frac{\phi_{t+1}}{\phi_t}, \\ c^n > 0, d^n > 0 \quad & \text{if} \quad \frac{1}{(1+\tau)} \frac{\psi_{t+1}}{\psi_t} = \frac{\phi_{t+1}}{\phi_t}. \end{aligned}$$

When $\frac{1}{(1+\tau)} \frac{\psi_{t+1}}{\psi_t} < \frac{\phi_{t+1}}{\phi_t}$, $d^n = 0$, and the sellers' participation constraint (4) and the first-order condition (27) gives the following equations.

$$x^n = \beta \frac{\phi_{t+1}}{\phi_t} c^n \tag{28}$$

$$u'(x^n) = \frac{1}{\beta} \frac{\phi_t}{\phi_{t+1}} \tag{29}$$

Then, solving (29) for x^n, and plug it into (28) yields,

$$\begin{aligned} x^n &= u'^{-1} \left(\frac{1}{\beta} \frac{\phi_t}{\phi_{t+1}} \right) \\ c^n &= \frac{1}{\beta} \frac{\phi_t}{\phi_{t+1}} u'^{-1} \left(\frac{1}{\beta} \frac{\phi_t}{\phi_{t+1}} \right) \end{aligned}$$

Following similar steps, when $\frac{1}{(1+\tau)}\frac{\psi_{t+1}}{\psi_t} > \frac{\phi_{t+1}}{\phi_t}$, $c^n = 0$, and

$$x^n = u'^{-1}\left(\frac{(1+\tau)}{\beta}\frac{\psi_t}{\psi_{t+1}}\right)$$
$$d^n = \frac{1}{\beta}\frac{\psi_t}{\psi_{t+1}}u'^{-1}\left(\frac{(1+\tau)}{\beta}\frac{\psi_t}{\psi_{t+1}}\right)$$

We can take similar steps to derive solutions to the case where $(1+\tau)\frac{\psi_{t+1}}{\psi_t} = \frac{\phi_{t+1}}{\phi_t}$, and furthermore to the next cases in monitored meetings.

Now, we set up the Lagrangian function for a buyer who will be monitored in the following DM as follows.

$$\mathcal{L}^m = u(x^m) - (1+\tau)c^m - (1+\tau)d^m + \lambda^m\left(\beta\frac{\phi_{t+1}}{\phi_t}c^m + \beta\frac{\psi_{t+1}}{\psi_t}d^m - x^m\right). \tag{30}$$

Then, the Kuhn-Tucker conditions are

$$\begin{aligned} u'(x^m) \le \lambda^m; \quad x^m \ge 0; \quad x^m\left(u'(x^m) - \lambda^m\right) = 0, \\ \lambda^m\beta\frac{\phi_{t+1}}{\phi_t} \le 1+\tau; \quad c^m \ge 0; \quad c^m\left(\lambda^m\beta\frac{\phi_{t+1}}{\phi_t} - 1 - \tau\right) = 0, \\ \lambda^m\beta\frac{\psi_{t+1}}{\psi_t} \le 1+\tau; \quad d^m \ge 0; \quad d^m\left(\lambda^m\beta\frac{\psi_{t+1}}{\psi_t} - 1 - \tau\right) = 0. \end{aligned}$$

We have the following first-order condition

$$u'(x^m) = \lambda^m. \tag{31}$$

Given the rates of return on cash and CBDC, $\frac{\phi_{t+1}}{\phi_t}$, and $\frac{\psi_{t+1}}{\psi_t}$, and the sales tax rate, τ, a buyer in the non-monitored DM meeting chooses

$$\begin{aligned} c^m > 0, d^m = 0 \quad \text{if} \quad \frac{\psi_{t+1}}{\psi_t} < \frac{\phi_{t+1}}{\phi_t}, \\ c^m = 0, d^m > 0 \quad \text{if} \quad \frac{\psi_{t+1}}{\psi_t} > \frac{\phi_{t+1}}{\phi_t}, \\ c^m > 0, d^m > 0 \quad \text{if} \quad \frac{\psi_{t+1}}{\psi_t} = \frac{\phi_{t+1}}{\phi_t} \end{aligned}$$

We can take the similar steps as above to obtain the solutions to each case. ■

Proof. Proof of Lemma 2.

Taking a partial derivative of T^s with respect to τ yields $\frac{\partial T^s}{\partial \tau} = \tau\{\rho\frac{\mu_d}{\beta}f'(\frac{\mu_d(1+\tau)}{\beta}) + (1+\rho)\frac{\mu_j}{\beta}f'(\frac{\mu_j(1+\tau)}{\beta})\} + \rho f(\frac{\mu_d(1+\tau)}{\beta}) + (1-\rho)f(\frac{\mu_j(1+\tau)}{\beta}) = 0$. Given $f'(\cdot) < 0$, if $f''(\cdot) > 0$ then there exists a unique upper bound $\tau = \bar{\tau} > 0$ by the Intermediate Value Theorem, because $-\tau\{\rho\frac{\mu_d}{\beta}f'(\frac{\mu_d(1+\tau)}{\beta}) + (1+\rho)\frac{\mu_j}{\beta}f'(\frac{\mu_j(1+\tau)}{\beta})\}$ increases from zero while $\rho f(\frac{\mu_d(1+\tau)}{\beta}) + (1-\rho)f(\frac{\mu_j(1+\tau)}{\beta})$ decreases to zero, when τ increases from zero. For example, assume a

CRRA utility function, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, where γ is a relative risk aversion parameter. Then, $f''(\cdot) = \frac{1+\gamma}{\gamma} x^{-\frac{1+2\gamma}{\gamma}} > 0$ holds and $\bar{\tau}$ is determined as $\frac{\gamma}{1-\gamma}$. ■

Proof. Proof of Lemma 3.

Given τ, and μ_j, taking a partial derivative of S with respect to μ_i yields $\frac{\partial S}{\partial \mu_i} = \frac{\rho}{\beta}\{\frac{(\mu_i - 1)}{\beta} f'(\frac{\mu_i}{\beta}) + f(\frac{\mu_i}{\beta})\} = 0$. Given $f'(\cdot) < 0$, if $f''(\cdot) > 0$ then there exists a unique upper bound $\mu_i = \bar{\mu}_i > 1$ by the Intermediate Value Theorem, because $-\frac{(\mu_i-1)}{\beta} f'(\frac{\mu_i}{\beta})$ increases from zero while $f(\frac{\mu_i}{\beta})$ decreases to zero, when μ_i increases from 1. For example, given the CRRA utility function, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $f''(\cdot) = \frac{1+\gamma}{\gamma} x^{-\frac{1+2\gamma}{\gamma}} > 0$ holds and $\bar{\mu}_i$ is determined as $\frac{1}{1-\gamma}$. The similar logic applies for μ_j with $\frac{\partial S}{\partial \mu_j} = \frac{(1-\rho)(1+\tau)}{\beta}\{\frac{(\mu_j-1)(1+\tau)}{\beta} f'(\frac{\mu_j(1+\tau)}{\beta}) + f(\frac{\mu_j(1+\tau)}{\beta})\} = 0$, and $\bar{\mu}_j = \frac{1}{1-\gamma}$ holds as well with the CRRA utility function, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$. ■

Proof. Proof of Proposition 1.

First, we will show that x^n increases as τ increases. Now suppose that μ_c is fixed when τ increases. We then have

$$\begin{aligned}
\frac{dG}{d\tau} &= (1-\rho)\left[\left((1-\gamma)u'(x^m) - \frac{1}{\beta}\right)\frac{dx^m}{d\tau} - (\frac{1-\beta}{\beta})\left(x^m + \tau\frac{dx^m}{d\tau}\right)\right] \\
&= (1-\rho)\left[\left((1-\gamma)u'(x^m) - \frac{1}{\beta}\right)\frac{u'(x^n)}{u''(x^m)} - (\frac{1-\beta}{\beta})\left(\frac{(1-\gamma)u'(x^m) - u'(x^n)}{u''(x^m)}\right)\right] \\
&= (1-\rho)\frac{1}{\beta u''(x^m)}\left[\left((1-\gamma)\frac{\mu_c}{\beta}(1+\tau) - \frac{1}{\beta}\right)\mu_c - (1-\beta)\left((1-\gamma)\frac{\mu_c}{\beta}(1+\tau) - \frac{\mu_c}{\beta}\right)\right] \\
&> 0
\end{aligned}$$

where the last inequality holds for $\beta > 1 - \frac{(1-\gamma)\mu_c(1+\tau)-1}{(1-\gamma)\mu_c(1+\tau)-\mu_c}$ since $(1-\gamma)\frac{\mu_c}{\beta}(1+\tau) - \frac{1}{\beta} = (1-\gamma)u'(x^m) - \frac{1}{\beta} < 0$ by Lemma 3. This implies μ_c must decrease as τ increases to maintain G in equilibrium. Hence, in equilibrium x^n increases as τ increases.

Second, in equilibrium $\frac{dx^m}{dx_n} \mid_G = -\frac{\rho\left[(1-\gamma)u'(x^n) - \frac{1}{\beta}\right]}{(1-\rho)\left[\left((1-\gamma)u'(x^m) - \frac{1}{\beta}\right) - \frac{1-\beta}{\beta}\tau\right]} < 0$ because $(1-\gamma)u'(x^k) - \frac{1}{\beta} < 0$ for $k \in \{n, m\}$ by Lemma 3. This implies, in equilibrium, x^m decreases as τ increases since x^n increases as τ increases. That is, equilibrium allocation (x^n, x^m) moves to lower-rightward when τ increases as in Figure 2.

Finally, note that $\frac{dx^m}{dx_n} \mid_W = -\frac{\rho(u'(x^n)-1)}{(1-\rho)(u'(x^m)-1)}$ and we have $\frac{dx^m}{dx_n} \mid_G = -\frac{\rho}{1-\rho} = \frac{dx^m}{dx_n} \mid_W < 0$ if $\tau = 0$. If $\tau > 0$, $\frac{dx^m}{dx_n} \mid_G < \frac{dx^m}{dx_n} \mid_W$ since $\frac{u'(x^n)-1}{u'(x^m)-1} < 1$ and $\frac{(1-\gamma)u'(x^n) - \frac{1}{\beta}}{\left((1-\gamma)u'(x^m) - \frac{1}{\beta}\right) - \frac{1-\beta}{\beta}\tau} = \frac{(1-\gamma)\frac{\mu_c}{\beta} - \frac{1}{\beta}}{\left((1-\gamma)\frac{\mu_c}{\beta}(1+\tau) - \frac{1}{\beta}\right) - \frac{1-\beta}{\beta}\tau} > 1$ for $\beta > 1-(1-\gamma)\mu_c$. That is, $\frac{dx^m}{dx_n} \mid_G \leq \frac{dx^m}{dx_n} \mid_W < 0$ for $\beta > 1-(1-\gamma)\mu_c$. This implies that social welfare decreases in τ, and, therefore, the optimal policy is $\tau = 0$ and $\mu_c = \hat{\mu}_c > 1$.

■

Proof. Proof of Proposition 2.

If the consolidated authority chooses τ and ζ, then μ_c is determined by the consolidated budget constraint (15). Given ζ, $\tau = 0$ is optimal according to Proposition 1. Now we will show given $\tau = 0$, $\zeta = 1$ is optimal. When $\tau = 0$, the consolidated authority budget constraint (16) can be written as

$$G = \rho x^n\{u^{'}(x^n) - \frac{1}{\beta}\} + (1-\rho)x^m\{u^{'}(x^m) - \frac{1}{\beta}\}. \tag{32}$$

This constraint and the FOC, $u^{'}(x^m) = \zeta u^{'}(x^n)$ determine the allocation (x^n, x^m). The slopes of the welfare function and the government budget constraint can be equal only at $x^n = x^m$ as follows:

$$\frac{\partial x_m}{\partial x_n}\mid_W = -\frac{\rho\{u^{'}(x^n) - 1\}}{(1-\rho)\{u^{'}(x^m) - 1\}} = -\frac{\rho\{(1-\gamma)u^{'}(x^n) - \frac{1}{\beta}\}}{(1-\rho)\{(1-\gamma)u^{'}(x^m) - \frac{1}{\beta}\}} = \frac{\partial x_m}{\partial x_n}\mid_{G_{\tau=0}},$$

Hence, given $\tau = 0$, $\zeta = 1$ is optimal. ■

Proof. Proof of Proposition 3.

If the consolidated authority chooses τ and ζ, then μ_d is determined by the consolidated budget constraint (17). Given ζ, $\tau = 0$ is optimal because the consolidated budget constraint (18) moves toward the origin symmetrically by raising τ. Therefore, both x^m and x^n must decrease when τ goes up. Given $\tau = 0$, $\zeta = 1$ is optimal by Proposition 2. Hence, $\tau = 0$ and $\zeta = 1$ is the optimal policy. μ_d is determined as $\hat{\mu}_c$ in (17), because (17) collapses into (15) with $\tau = 0$. ■

Proof. Proof of Lemma 4.

Since the CBI is required, given $G > 0$, τ and $\zeta\mu_c$ must satisfy with (19) in equilibrium. Given $\zeta\mu_c$, $\frac{\partial T^s}{\partial \tau} > 0$ in $\tau \in [0, \tau)$ by Lemma 2. Given τ, $\frac{\partial T^s}{\partial \zeta\mu_c} < 0$ in (19) because $f^{'}(\cdot) < 0$. By the Implicit Function Theorem, $\frac{d\zeta\mu_c}{d\tau} = -\frac{\frac{\partial T^s}{\partial \tau}}{\frac{\partial T^s}{\partial \zeta\mu_c}} > 0$ in (19). ■

Proof. Proof of Proposition 4.

Given ζ, both x^n and x^m decrease in μ_c by the FOCs, $u^{'}(x^n) = \frac{\mu_c}{\beta}$ and $u^{'}(x^n) = \frac{\mu_c(1+\tau)}{\beta}$. Thus, $\frac{\partial W}{\partial \mu_c} < 0$. Since $S \geq 0$ in (20), $\mu_c = 1$ is required for optimality. Then, $\tau = \hat{\tau} > 0$, where $\hat{\tau}$ solves $G = (1-\rho)\hat{\tau}f(\frac{1+\hat{\tau}}{\beta})$ in (19). ■

Proof. Proof of Proposition 5.

Given the non-negative seigniorage condition (22), the optimal allocation in a cash and CBDC economy is $x^n = x^m$ at point E in Figure 3 according to Proposition 2. Since $x^n = x^m$ implies that $\mu_c^* = \zeta\mu_c^*(1+\tau)$ according to Lemma 1, ζ must satisfies with $\zeta(1+\tau^*) = 1$. Then, μ_c^* is determined by (21), $G = (\mu_c^* - 1)f\left(\frac{\mu_c^*}{\beta}\right)$, with $\tilde{\mu} = \mu_c$ and $\zeta(1+\tau^*) = 1$. Finally, τ^* is determined by $G = (1-\rho)\tau^* f\left(\frac{\mu_c^*}{\beta}\right)$ in (19). In addition, when $\zeta(1+\tau)$ decreases in (21), μ_c must increase, and $\tilde{\mu}$ decrease. As a result, $\tau^* < \hat{\tau}$ because $\tilde{\mu}$ decreases as $\zeta(1+\tau)$ falls down from $1+\tau$ to 1. ■

Proof. Proof of Proposition 6.

Given the non-negative seigniorage condition (22), the optimal allocation in a cash and CBDC economy is $x^n = x^m$ at point E in Figure 3 as shown in Proposition 2. Since $x^n = x^m$ implies that $\mu_d(1+\tau) = \zeta\mu_d(1+\tau)$ according to Lemma 1, $\zeta = 1$ is optimal. Then, $\tilde{\mu}_d$ is determined by $G = (\tilde{\mu}_d - 1) f\left(\frac{\tilde{\mu}_d}{\beta}\right)$ in (25) and $\tau = \tau^{**}$ is also pinned down by $G = \tau^{**} f\left(\frac{\tilde{\mu}_d}{\beta}\right)$ from (23). By comparing with $G = (\mu_c^* - 1) f\left(\frac{\mu_c^*}{\beta}\right)$ and $G = (1-\rho)\tau^* f\left(\frac{\mu_c^*}{\beta}\right)$ in the proof of Proposition 5, we know that $\tilde{\mu}_d = \mu_c^*$, and thus $0 < \tau^{**} < \tau^*$. Since $\tilde{\mu}_d = \mu_d(1+\tau^{**})$ in equilibrium, $\mu_d = \hat{\mu}_d < \mu_c^*$. ■